A REMARK "ON THE PROJECTIVE MOTION IN A PROJECTIVE FINSLER SPACE OF RECURRENT CURVATURE"

By

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The present paper is in the continuation of the author's previous paper [1][1] in which we have dealt with the projective motion in a projective Finsler Space under recurrence property. However, there is much to be explored and therefore this paper has been designed to reconsider some particular cases in a more general view. All the notations and symbolism in the current discussion stand the same as in [1].

In a projective Finsler space $\mathfrak{F}_n$ with normal projective connection $\pi^i_{jk}(x, \dot{x})$, let us consider an infinitesimal transformation $\bar{x}^i = x^i + \xi^i(x)dt$ with respect to a contravariant vector field ξ^i (which is only a point function). For this transformation being a projective motion in an $\mathfrak{F}_n$-space, we have [3]

$$\underset{L}{D}\pi^i_{jk} = \delta^i_j \varphi_k + \delta^i_k \varphi_j \, , \tag{1}$$

$$\underset{L}{D} W^i_{hjk} = 0 \, , \tag{2}$$

where $\underset{L}{D}$ denotes the operator of Lie differentiation, φ_j is a covariant vector and $W^i_{hjk}(x, \dot{x})$ is the Weyl's projective curvature tensor [2, 3], which satisfies the following relations:

$$\begin{cases} \text{a)} \quad W^l_{ljk} = W^l_{hlk} = W^l_{hjl} = 0, \quad \dot{\partial}_i W^i_{hjk} = 0, \quad \dot{x}_l \dot{\partial}_l W^i_{hjk} = 0 \, , \\ \text{b)} \quad W^i_{hjk}\dot{x}^h = W^i_{jk}, \quad W^i_{hjk}\dot{x}^h \dot{x}^j = W^i_k, \quad W^i_k \dot{x}^k = 0, \quad \dot{\partial}_i W^i_k = 0 \, . \end{cases} \tag{3}$$

In a projective Finsler space $\mathfrak{F}_n$, the corresponding normal projective curvature tensor $N^i_{hjk}(x, \dot{x})$ of $\mathfrak{F}_n$ with respect to the normal projective connection $\pi^i_{jk}(x, \dot{x})$ is defined by

$$N^i_{hjk} = \partial_h \pi^i_{jk} - \partial_j \pi^i_{hk} - \pi^l_{hr}\dot{x}^r \dot{\partial}_l \pi^i_{jk} + \pi^l_{jr}\dot{x}^r \dot{\partial}_l \pi^i_{hk} + \pi^i_{hl}\pi^l_{jk} - \pi^i_{jl}\pi^l_{hk} \, ,$$

where

$$\partial_h \equiv \frac{\partial}{\partial x^h} \, , \quad \dot{\partial}_l \equiv \frac{\partial}{\partial \dot{x}^l} \, .$$

If the normal projective curvature tensor N^i_{hjk} of the space $\mathfrak{F}_n$ satisfies the

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1) Numbers in brackets refer to the references at the end of this paper.

relation $N^i_{hjk/l}=\lambda_l N^i_{hjk}$ for a non-zero covariant vector λ_l, the space $\mathfrak{F}_n$ is called a projective Finsler space of recurrent curvature denoted by $\mathfrak{F}^{\oplus}_n$. In an $\mathfrak{F}^{\oplus}_n$-space, we have the relation $W^i_{hjk/l}=\lambda_l W^i_{hjk}$ [1]. Hence, in virtue of this and (2), we can find with ease

$$\underset{L}{D} W^i_{hjk/l}=(\underset{L}{D}\lambda_l) W^i_{hjk} . \tag{4}$$

We now recall the formula

$$\underset{L}{D}(W^i_{hjk/l})-(\underset{L}{D}W^i_{hjk})_{/l}=(\underset{L}{D}\pi^i_{lm})W^m_{hjk}-(\underset{L}{D}\pi^m_{lh})W^i_{mjk}-(\underset{L}{D}\pi^m_{lj})W^i_{hmk}$$
$$-(\underset{L}{D}\pi^m_{lk})W^i_{hjm}-(\underset{L}{D}\pi^m_{ls})\dot{x}^s\dot{\partial}_m W^i_{hjk} .$$

Substitution of (1) and (4) into the above formula and the use of condition (2) gives [1]

$$(\underset{L}{D}\lambda_l)W^i_{hjk}=\delta^i_l\varphi_m W^m_{hjk}-2\varphi_l W^i_{hjk}-\varphi_h W^i_{ljk}-\varphi_j W^i_{hlk}-\varphi_k W^i_{hjl}-\varphi_s\dot{x}^s\dot{\partial}_l W^i_{hjk} \tag{5}$$

In my previous paper [1], we have already proved the following theorem:

Theorem. *When an $\mathfrak{F}^{\oplus}_n$-space ($n\geqq 3$) admits an infinitesimal projective motion satisfying $\varphi_m W^m_{hjk}\neq 0$, then the following relation holds:*

$$\underset{L}{D}\lambda_l=(n-2)\varphi_l . \tag{6}$$

Now we devote ourselves in discussing through some particular cases more generally.

I. The case of $\varphi_m W^m_{hjk}\neq 0$. Substituting (6) into (5), we have

$$n\varphi_l W^i_{hjk}=\delta^i_l\varphi_m W^m_{hjk}-\varphi_h W^i_{ljk}-\varphi_j W^i_{hlk}-\varphi_k W^i_{hjl}-\varphi_s\dot{x}^s\dot{\partial}_l W^i_{hjk} . \tag{7}$$

We now suppose that u^l is any contravariant vector. Contracting (7) with $\varphi_i u^l u^h u^j$, making use of (3a) and finally we put $u^l=\dot{x}^l$ etc. in the result, then in virtue of (3a) and (3b) we can obtain

$$\varphi=0 \quad \text{or} \quad \varphi_i W^i_k=0, \quad \text{where} \quad \varphi\equiv\varphi_l\dot{x}^l ,$$

because of $\varphi(x, \dot{x})$ is an arbitrary scalar function positively homogeneous of the first degree in $\dot{x}^i$ and $\varphi_i\equiv\partial\varphi/\partial\dot{x}^i$. The first case indicates that the motion is affine, while we see that the second is a consequence of $\varphi_m W^m_{hjk}=0$, and on account of our assumption, conclusively this can be excluded. Thus we have

Theorem 1. *If an $\mathfrak{F}^{\oplus}_n$-space ($n\geqq 3$) admits the infinitesimal projective motion satisfying $\varphi_m W^m_{hjk}\neq 0$, then the motion is necessarily an affine one.*

II. The case of $\varphi_m W^m_{hjk}=0$. In the present case, the equation (5) becomes

$$(\underset{L}{D}\lambda_l+2\varphi_l)W^i_{hjk}=-\varphi_h W^i_{ljk}-\varphi_j W^i_{hlk}-\varphi_k W^i_{hjl}-\varphi_s\dot{x}^s\dot{\partial}_l W^i_{hjk} .$$

We again assume that u^i is any contravariant vector. Contracting the above equation with $u^iu^hu^j$ and simplifying over it by putting $u^i=\dot{x}^i$ etc., then in virtue of (3), we can get at last

$$(8) \qquad \text{a)} \quad (\underset{L}{D}\lambda_i+4\varphi_i)\dot{x}^i=0, \quad \text{or} \quad \text{b)} \quad W^i_k=0\,.$$

Moreover, in the case of (b), since W^i_k vanishes, therefore, accordingly, the tensor W^i_{hjk} also vanishes identically.[2] Here we notice that when the generalized Weyl's projective curvature tensor $W^i_{hjk}(x, \dot{x})$ of the projective Finsler space $\mathfrak{F}_n$ is zero throughout the space, then we call the space $\mathfrak{F}_n$ to be projectively flat one, and therefore, in the second case, our space is a projectively flat space. Thus we have

Theorem 2. *If an $\mathfrak{F}_n$-space $(n\geqq 3)$ admits the infinitesimal projective motion satisfying $\varphi_m W^m_{hjk}=0$, then one of the following two conditions must be satisfied*: (1) *The space is a projectively flat one.* (2) *The motion must satisfy the relation $(\underset{L}{D}\lambda_i+4\varphi_i)\dot{x}^i=0$.*

Further from theorems 1 and 2, we can obtain

Theorem 3. *If an $\mathfrak{F}^{\oplus}_n$-space $(n\geqq 3)$ admits the infinitesimal projective motion which is not an affine one, then the relation $\varphi_m W^m_{hjk}=0$ always holds good and one of the following two conditions must be satisfied*: (1) *The space is a projectively flat space,* (2) *The motion must satisfy the relation $(\underset{L}{D}\lambda_i+4\varphi_i)\dot{x}^i=0$.*

From above theorems, we may also have the following:

Theorem 4. *If an $\mathfrak{F}^{\oplus}_n$-space $(n\geqq 3)$ which is a projectively non-flat, admits the infinitesimal projective motion satisfying $(\underset{L}{D}\lambda_i+4\varphi_i)\dot{x}^i\neq 0$, then the motion is necessarily an affine one.*

Theorem 5. *If an $\mathfrak{F}^{\oplus}_n$-space $(n\geqq 3)$ admits the infinitesimal projective motion which is not affine and $(\underset{L}{D}\lambda_i+4\varphi_i)\dot{x}^i\neq 0$, then the space is a projectively flat one.*

After this discussion, on one hand, from (6) and (8a), a one of the theorem of our previous paper [1] can more coherently be stated as follows:

Theorem 6. *In order that an infinitesimal projective motion admitted in an $\mathfrak{F}^{\oplus}_n$-space $(n\geqq 3)$ which is a projectively non-flat, become an affine one, it is necessary and sufficient that $\underset{L}{D}\lambda_i=0$.*

On the other hand, from the theorems 2, 5 and in virtue of the theorem of

[2] H. Rund [4], p. 142.

Berwald[3], we can give the

Theorem 7. *If an* $\mathfrak{F}_n^{\oplus}$*-space* ($n \geqq 3$) *admits the infinitesimal projective motion (which is not affine) satisfying* $\varphi_m W^m_{hjk}=0$ *and* $(\underset{L}{D}\lambda_l+4\varphi_l)\dot{x}^l \neq 0$, *then a general path space of n dimensions is mapped by means of a projective change onto a general path space of zero curvature* ($H^i_{jkl}=0$).

We also remember the theorem[4]: 'The generalized Weyl tensor vanishes identically in an isotropic Finsler space'. By reason of this theorem, we can ennunciate the

Theorem 8. *If an* $\mathfrak{F}_n^{\oplus}$*-space* ($n \geqq 3$) *admits the infinitesimal projective motion (which is not affine) satisfying* $\varphi_m W^m_{hjk}=0$ *and* $(\underset{L}{D}\lambda_l+4\varphi_l)\dot{x}^l \neq 0$, *then the space is an isotropic Finsler space of recurrent curvature.*

This completes our discussion.

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3) L. Berwald 2, p. 767.

4) H. Rund [4], p. 147.