ORIENTABLE 3-MANIFOLDS AS SINGULAR BLOCK BUNDLES

By

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1. Introduction

We introduced the concept of singular block bundles over fake manifolds in [2]. And the following theorem is already proved there.

Theorem. *Let V be a 3-manifold with non-empty boundary and P a closed fake surface which is a spine of V. Then, V is a singular block bundle over P with fiber-set Φ^1, that is, V belongs to $B^1_2(P)$.*

Throughout this paper, we use the definitions and notations in [1] and [2]. And, let $B(P)$ and $+B(P)$ denote the set $B^1_2(P)$ and the subset of $B(P)$ consisting of orientable 3-manifolds, respectively, for a fake surface P with $\mathfrak{S}_7(P)=\emptyset$ (for the numbering of the singularities of P, we use the definition made in [2]).

The main purpose of this paper is to count the number of the elements of the set $+B(P)$ for a given closed fake surface P.

Theorem 1. *Let P be a closed fake surface and put $\lambda=\#\dot{U}(P)-\#M(P)$. Then, we obtain*

$$\#(+B(P))\leqq 2^{\lambda} .$$

Especially, we obtain the following.

Theorem 2. *Let P be a closed fake surface with $H_1(P)=0$ and $+B(P)\neq\emptyset$. Then, we obtain*

$$\#(+B(P))=2^{(\#U(P)-1)} .$$

In § 2, we study about $B(U(P))$ for a closed fake surface P. That is, we show that $B(U(P))$ consists of exactly one element for any closed fake surface P. Furthermore, it is shown that any equivalence of the element of $B(U(P))$ is isotopic to the identity keeping $U(P)$ fixed.

In § 3, first, we show that, for 2-manifold M, $+B(M)$ consists of exactly one element. Next, we show that $B(P)$ consists of exactly one element if P is a closed fake surface such that $M(P)$ consists of 2-balls and $+B(P)\neq\emptyset$. Finally, it is known that $+B(P)$ consists of at most one element if $U(P)$ is connected.

In § 4, we prove Theorem 1 and Theorem 2.

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2. Propositions about $B(U(P))$

First of all, we prove the following.

Proposition 1. *Let P be a closed fake surface. Then, $B(U(P))$ consists of exactly one element.*

Proof. We may assume that $U(P)$ is connected.

Step 1. We show the existence of an element of $B(U(P))$.

Case 1. Suppose that $\mathfrak{S}_3(P)$ is empty. Then, $U(P)$ is either $S\times T$ or $S\times_\sigma T$ or $S\times_\tau T$, by Lemma 5 [1]. Let us consider the pair $(D, T)\times I$, where D is a 2-ball which contains a T-shaped 1-polyhedron T properly and I denotes the closed interval $[0, 1]$. It is not hard to see that $D\times I$ is a singular block bundle over $T\times I$ with $(D\times I|T\times 0)=D\times 0$ and $(D\times I|T\times 1)=D\times 1$. Let h denote the homeomorphism from $T\times 0$ onto $T\times 1$ such that $(T\times I)/h$ is the given $U(P)$. Then, h can be extended to a homeomorphism H from $D\times 0$ onto $D\times 1$ so that $\eta=(D\times I)/H$ is a singular block bundle over $U(P)$ which is clearly a required element of $B(U(P))$.

Case 2. Suppose that $\mathfrak{S}_3(P)$ is non-empty. Then, we can write

$$U(P)=\bigcup_x N_x \cup \bigcup_j (T\times I)_j\,,$$

where $N_x=\mathrm{st}(x, U(P))$ with x in $\mathfrak{S}_3(P)$ and $(T\times I)_j$ denotes a closure of a connected component of $U(P)-\cup N_x$, let us consider the standard pairs (B_x, N_x) and $(D_j, T_j)\times I$ where $T_j\times I=(T\times I)_j$, (for the standard pairs, see [2]). Suppose that $N_x\cap(T\times I)_j=(T\times 0)_j$ and h_{jx} denotes the identification map from $(T\times I)_j$ to N_x. Note that B_x and $D_j\times I$ are singular block bundles over N_x and $T_j\times I$, respectively. It is not hard to obtain a homeomorphism H_{jx} from $D_j\times 0$ onto $(B_x|(T\times 0)_j)$ extending h_{jx} so that $(D_j\times I\cup B_x)/H_{jx}$ is a singular block bundle over $(T_j\times I\cup N_x)/h_{jx}$. Continueing the above process, we obtain an element η of $B(U(P))$.

Step 2. Here, we have to prove the uniqueness of the element η of $B(U(P))$. We prove just the case that $\mathfrak{S}_3(P)$ is non-empty, because we can prove the case that $\mathfrak{S}_3(P)$ is empty by a similar argument. We use the representation of $U(P)$ written in Step 1. Put $B_x=(\eta|N_x)$. Then, $(\eta|(T\times I)_j)$ can be considered as a 1-handle W_j attached to the disjoint union $\cup B_x$ of 3-balls by the homeomorphism

H_{jx} from $(W_j|(T\times\dot{I})_j)$ onto $(\cup B_x|(T\times\dot{I})_j)$. Note that H_{jx} is determined only by $H_{jx}|(T\times\dot{I})_j$ up to isotopy. Thus, it is easy to see that η is unique. This completes the proof of Proposition 1.

Corollary to Proposition 1. (1) *$B(U(P))$ consists of a solid torus with genus 1, if $U(P)$ is either $S\times\sigma T$ or $S\times T$.*

(2) *$B(U(P))$ consists of a solid Klein bottle with genus 1, if $U(P)$ is $S\times\tau T$.*

Next, we state a propesition about equivalences of the unique element η of $B(U(P))$ for a closed fake surface P.

Proposition 2. *Let P be a closed fake surface, and η the element of $B(U(P))$. Suppose that h is an equivalence of η onto itself. Then, h is isotopic to the identity keeping $U(P)$ fixed.*

In order to prove the above proposition, we need some lemmas about equivalences of the standard pairs.

Suppose that η is an element of $B_p^n(P)$ and H an isotopy of η, that is, H is a level-preserving homeomorphism from $\eta\times I$ onto $\eta\times I$. Note that $\eta\times I$ can be regarded as an element of $B_p^{n+1}(P)$ by the natural way. Then, we say that H is a *block-preserving isotopy* of η, if H is an equivalence of $\eta\times I$ as a singular block bundle.

Lemma 1. *Let $(D, T)\times I$ denote the standard pair. Suppose that h is an equivalence of $D\times I$ onto itself. Then, there exists a block-preserving isotopy of $D\times I$ sending h to the identity.*

Proof. *Step 1.* Here, we consider $h_0=h|D\times 0$. It is clear that $D\times 0=(D\times I|T\times 0)$ is a singular block bundle over $T\times 0$ and h_0 is an equivalence of $D\times 0$. We write $D=D\times 0$ and $T=T\times 0$. Let $D_1, \cdots, D_6$ denote the closures of the connected components of $D-(T\cup F_{o(T)})$, where $F_{o(T)}$ means the block of $D\times I$ over $o(T)$. Since $h_0|T$ is the identity, so is with $h_0|(F_{o(T)})^{\cdot}$. Thus, $h_0|F_{o(T)}$ is isotopic to the identity keeping $o(T)$ fixed by an isotopy G_1 of $F_{o(T)}$. Hence, we can extend G_1 to an isotopy G_{1i} of D_i so that G_{1i} sends $h_0|D_i$ to the identity and keeps $D_i\cap T$ fixed, because h_0 keeps D_i set-wise fixed. Combining G_{1i}, we obtain a block-preserving isotopy G_2 of D sending h_0 to the identity. Then, it is not hard to extend G_2 to a block preserving isotopy H_0 of $D\times I$ sending h_0 to the identity.

Step 2. By Step 1, we can assume that $h|D\times\dot{I}$ is the identity. Let us consider, first the closures of the connected components of $(T\times I)-o((T)\times I)$ and second, the ones of $D\times I-(D\times I|o(T)\times I)$. Then, by the same way as Step 1, we obtain a required block-preserving isotopy of $D\times I$, because h keeps each of the above closures set-wise fixed.

By a similar argument to Lemma 1, we obtain the following lemma.

Lemma 2. *Let (B, St_3) denote a standard pair. Suppose that h is an equivalence of B onto itself. Then, there exists a block-preserving isotopy of B sending h to the identity.*

Now, we prove Proposition 2.

Proof of Proposition 2. We can write

$$U(P)=\bigcup_{x} \operatorname{st}(x, U(P)) \cup \bigcup_{j} (T\times I)_j ,$$

as in [1], where x ranges over $\mathfrak{S}_3(P)$. Then, it is easy to see that $(\eta|\operatorname{st}(x, U(P))$ and $(\eta|(T\times I)_j)$ are standard pairs (B, St_3) and $(D, T)\times I$, respectively. Since $h|(\eta|\operatorname{st}(x, U(P))$ and $h|(\eta|(T\times I)_j)$ satisfy the conditions of Lemma 2 and Lemma 1. Thus, we obtain a required isotopy of η sending h to the identity keeping $U(P)$ fixed. This completes the proof of Proposition 2.

Remark. As is seen in the proof, the isotopy required in Proposition 2 can be chosen to be block-preserving.

3. Orientable 3-manifolds as singular block bundles

First of all, we prove the following proposition.

Proposition 3. *Let M be a 2-manifold. Then, $+B(M)$ consists of exactly one element.*

Proof. *Step 1.* We construct an element η in $+B(M)$ which is an orientable 3-manifold.

Case 1. Suppose that $\dot{M}$ is non-empty. Then, we can regard M as a 2-ball B with bands B_i, $i=1, \cdots, n$, such that, putting $B_i=C_i\times J$ with C_i a 1-ball and J the closed interval $[-1, 1]$, we have the following conditions.

(1) $B_i\cap B_j=\emptyset$, if $i\neq j$.

(2) $B_i\cap B=\dot{B}_i\cap \dot{B}=\dot{C}_i\times J$.

Now, let us consider the 3-balls $\tilde{B}=B\times J$ and $\tilde{B}_i=B_i\times J$. Then, it is not hard to see that there exists an identification map h_i from $\dot{C}_i\times J\times J$ onto itself so that the block bundle $(\tilde{B}\cup\tilde{B}_i)/h_i$ over $B\cup B_i$ is a solid torus with genus 1 for any i. Thus, we obtain an element

$$\eta=\bigcup_{i} (\tilde{B}\cup\tilde{B}_i)/h_i ,$$

in $+B(M)$.

Case 2. Suppose that M is a closed 2-manifold. Let A be a 2-simplex of M and $M_1=M-\mathring{A}$. Then, by Case 1, we have an element η_1 in $+B(M_1)$. Put $\eta_2=A\times J$ with $A=A\times 0$. Note that both $(\eta_1|\dot{A})$ and $(\eta_2|\dot{A})$ are bands because η_1 and η_2 by identifying $(\eta_1|\dot{A})$ and $(\eta_2|\dot{A})$ by a suitable homeomorphism.

Step 2. Here, we show the uniquness of η of $+B(M)$. We prove just the case when $\dot{M}$ is non-empty, because it is not hard to prove the case when M is closed. Suppose that η_1 and η_2 are elements of $+B(M)$. Put $(\eta_j|B)=\tilde{B}^j$ and $(\eta_j|B_i)=\tilde{B}_i^j$, where B and B_i denote the 2-ball and the bands described in Step 1. Since both B and B_i are 2-balls, we can write $\tilde{B}^j=B\times J$ and $\tilde{B}_i^j=B_i\times J$. And, furthermore, we see $\tilde{B}^j\cap\tilde{B}_i^j=(\eta_j|\dot{C}_i\times J)$ where $B_i=C_i\times J$ as in Step 1. Then, by checking the identification maps from $\tilde{B}_i^j$ to $\tilde{B}_j$, it is known that η_1 and η_2 are equivalent, by making use of the fact that $\tilde{B}^j\cup\tilde{B}_i^j$ is a solid torus with genus 1. Thus, Proposition 3 is established.

Lemma 3. *Let P be a closed fake surface such that $M(P)$ consists of 2-balls. Then, $B(P)$ consists of at most one element.*

Proof. Suppose that there exist two elements η_1 and η_2 in $B(P)$. We have to show that η_1 and η_2 are equivalent. By Proposition 1, there exists an equivalence h_U from $(\eta_1|U(P))$ to $(\eta_2|U(P))$. Now, $(\eta_i|M)$, $i=1, 2$, is equivalent to $M\times J$ ($J=[-1, 1]$), because M is a 2-ball for any M of $M(P)$. Then, we can extend $h_U|(\eta_1|\dot{M})$ to an equivalence h_M from $(\eta_1|M)$ to $(\eta_2|M)$. Thus, it is easy to obtain a required equivalence from η_1 to η_2 defined by h_U and h_M.

Lemma 4. *Let P be closed fake surface with $\#\mathfrak{S}_2(P)=1$ ($\#$ means the number of the connected components). Then, $+\dot{B}(P)$ consists of at most one element.*

Proof. Let η_1 and η_2 are elements of $+B(P)$ and h_U an eqivalence from $(\eta_1|U(P))$ to $(\eta_2|U(P))$. Let M be an element of $M(P)$. If M is a 2-ball, we can extend $h_U|(\eta_1|\dot{M})$ to an equivalence h_M from $(\eta_1|M)$ to $(\eta_2|M)$ as in Lemma 3. Thus, we assume that M has non-empty boundary and is not a 2-ball. Then, there exist disjoint proper 1-balls $A_1, \cdots, A_m$ in M such that the closure B of $M-\bigcup_{i=1}^{m} N_i$ is a 2-ball, where N_i means the 2-nd derived neighborhood of A_i in M. Since both η_1 and η_2 are orientable, it is not hard to see that h_U can be extended to an equivalence from $(\eta_1|U(P)\cup\cup N_i)$ to $(\eta_2|U(P)\cup\cup N_i)$ which is denoted by h_A. Then, by the same reason as in the proof of Lemma 3, $h_A|\dot{B}$ can be extended to equivalence h_B from $(\eta_1|B)$ to $(\eta_2|B)$. Thus, we obtain a required equivalence from η_1 to η_2 by h_A and h_B.

Lemma 5. *Let η_1 and η_2 be elements of $+B(P)$ for a closed fake surface P. Then, η_1 and η_2 are equivalent if and only if there exist orientations of η_1 and η_2 such that an equivalence from $(\eta_1|U(P))$ to $(\eta_2|U(P))$ is orientation preserving.*

Proof. The necessity is trivial. So, we prove just the sufficiency. Let the equivalence from $(\eta_1|U(P))$ to $(\eta_2|U(P))$ given in the hypothesis be h_U. The proof goes by induction on $u=\#U(P)$. When $u=1$, Lemma 4 gives the answer. Suppose $u\geqq 2$.

Step 1. Let α denote a 1-ball in P satisfying the following three conditions.

(1) $\alpha\cap\mathfrak{S}_2(P)=\dot{\alpha}$.

(2) $\alpha\cap\mathfrak{S}_3(P)=\emptyset$.

(3) There exist two (distinct) connected components of $\mathfrak{S}_2(P)$ which intersect with α.

Then, it is not hard to see that h_U can be extended to an equivalence from $(\eta_1|U(P)\dot{\cup} N)$ to $(\eta_2|U(P)\cup N)$ where N means the 3-rd derived neighborhood of α in P.

Step 2. Let us consider a fake surface N_o defined as follows. In R^3, put

$A_1=\{(1, y, z)\,|\,|y|, |z|\leqq 1\}$,

$A_2=\{(-1, y, z)\,|\,|y|, |z|\leqq 1$ and either $|y|\geqq 1/2$ or $|z|\geqq 1/2\}$.

$A_3=\{(x, 0, z)\,|\,|x|, |z|\leqq 1$ and $|z|\geqq 1/2\}$.

$A_4=\{(x, y, z)\,|\,|x|\leqq 1$, and either $|y|\leqq 1/2, |z|=1/2$ or $|y|=1/2, |z|\leqq 1/2\}$.

Define N_0 to be the union of $A,\cdots,A_4$. Then, it is not hard to see that $+B(N_0)$ consists of exactly one element η_0. Now, we define a closed fake surface P' to be the union of $P-\mathring{N}$ and N_0 such that the natural union η_i' of $\eta_i|(P-\mathring{N})$ and η_0 is an element of $+B(P')$. It is known easily that $\#U(P')\leqq u-1$ and there exists an orientation preserving equivalence from $(\eta_1'|U(P'))$ to $(\eta_2'|U(P'))$. Then, by the inductive hypothesis, there exists an equivalence h' from η_1' to η_2'.

Step 3. The result in Step 2 implies that $(\eta_1|P-\mathring{N})$ and $(\eta_2|P-\mathring{N})$ are equivalent by the restriction $h|(\eta_1'|P-\mathring{N})$. The rest of the proof is easy, because we can extend $h|(\eta_1|\dot{N})$ to an equivalence from $(\eta_1|N)$ to $(\eta_2|N)$.

4. Theorems

In this section, we prove the theorems stated in the introduction.

Theorem 1. *Let P be a closed fake surface with $\mathfrak{S}_2(P)\neq\emptyset$ and put $\lambda=\#\dot{U}(P)-\#M(P)$. Then, we obtain*

$$\#(+B(P))\leqq 2^{\lambda}.$$

Proof. Put $u=\#U(P)$. When $u=1$, we see the conclusion by Lemma 4. On the other hand, suppose $\lambda=0$. Then, $M(P)$ consists of 2-balls. Thus, Theorem 1 holds for $\lambda=0$ by Lemma 3. Here, let us define the order of the pair (u, λ) by $(u, \lambda)>(u', \lambda')$ if and only if either $u>u'$ or $u=u'$ and $\lambda>\lambda'$. It is sufficient to prove Theorem 1 for (u, λ) assuming Theorem 1 for (u', λ') with $(u, \lambda)>(u', \lambda')$. We deal with the case $u\geqq 2$ and $\lambda\geqq 1$. Then, there exists an element M of $M(P)$ with $\#\dot{M}\geqq 2$. Let b denote a boundary component of M. Then, the derived neighborhood of b in P is a band. And, for any element η of $+B(P)$, $(\eta|b)$ is also a band. Thus, b disconnects P if and only if $(\eta|b)$ disconnects η.

Case 1. Suppose that b does not disconnect P. Let P_1 and η_1 denote the fake surface and singular block bundle obtained from P and η by cutting them along b and $(\eta|b)$, respectively. Let b_1 and b_2 denote the two copies of b which are the boundary components of P_1. We can construct a closed fake surface $\tilde{P}$ from P_1 by attaching two 2-balls to b_1 and b_2. Similarly, we have a natural singular block bundle $\tilde{\eta}$ over P from η_1 by attaching 2-handles to $(\eta_1|b_1)$ and $(\eta_1|b_2)$. Of course, $\tilde{\eta}$ is an element of $+B(\tilde{P})$. On the other hand, we obtain an element of $+B(P)$ from η_1 by identifying $(\eta_1|b_1)$ and $(\eta_1|b_2)$ by an equivalence uniquely, for η_1 is connected. Thus, we have $\#(+B(P))\leqq\#(+B(\tilde{P}))$. And, it is not hard to see

$$\lambda=\#\dot{U}(P)-\#M(P)>\#\dot{U}(\tilde{P})-\#M(\tilde{P})=\lambda'.$$

Hence, by the inductive hypothesis, we obtain

$$\#(+B(P))\leqq\#(+B(P))\leqq 2^{\lambda'}<2^{\lambda}.$$

Case 2. Suppose that b disconnects P into two fake surfaces P_1 and P_2. Consequently, $(\eta|b)$ disconnects η into η_1 and η_2 which are singular block bundles over P_1 and P_2, respectively. Then, by the same reason as above, we can regard η_i to be the singular block bundle obtained by restricting some element of $+B(\tilde{P}_i)$, where $\tilde{P}_i$ is the closed fake surface obtained from P_i as in Case 1, $i=1, 2$. Now, in this case, there exist two isotopy classes of equivalences h from $(\eta_1|b_1)$ to $(\eta_2|b_2)$ such that $(\eta_1\cup\eta_2)/h$ is an element of $+B(P)$. Thus, we obtain

$$\#(+B(P))\leqq 2\times\#(+B(\tilde{P}_1))\times\#(+B(\tilde{P}_2)).$$

Since $\#\dot{M}\geqq 2$, it is clear that $\#U(P_i)\leqq u-1$. Then, by the inductive hypothesis, we have $\#(+B(P_i))\leqq 2^{\lambda_i}$, where $\lambda_i=\#\dot{U}(\tilde{P}_i)-\#M(\tilde{P}_i)$. Thus, $\#(+B(P))=2^{\lambda_1+\lambda_2+1}$ follows directly. On the other hand, we see

$$\begin{aligned}\lambda &= \#\dot{U}(P)-\#M(P)\\ &=(\#\dot{U}(\tilde{P}_1)+\#\dot{U}(\tilde{P}_2))-(\#M(\tilde{P}_1)+\#M(\tilde{P}_2)-1)\\ &=\lambda_1+\lambda_2+1\ .\end{aligned}$$

Hence, we obtain the required result $\#(+B(P))\leqq 2^{\lambda}$.

In the following, we consider about $+B(P)$ for a closed fake surface P with $H_1(P)=0$.

Lemma 6. *Let P be a closed fake surface with $H_1(P)=0$. Then, we obtain*

$$\#\dot{U}(P)-\#M(P)=\#U(P)-1\ .$$

Proof. When $U(P)$ is empty, the result is trivial, because P must be a 2-sphere. And if $u=\#U(P)=1$, there is nothing to do, for $M(P)$ consists of 2-balls by [1]. So, we assume $u\geqq 2$. Then, take an element M of $M(P)$ with $\#\dot{M}\geqq 2$ and let b denote a boundary component of M. Since $H_1(P)=0$, b disconnects P into two fake surfaces P_1 and P_2. Let $\tilde{P}_i$ denote the closed fake surfaces obtained from P_i by attaching 2-balls to their boundary, $i=1,2$. It is not hard to see $H_1(\tilde{P}_i)=0$ and $\#U(P_i)\leqq u-1$ for both $i=1,2$. Thus, by the inductive hypothesis, we obtain

$$\#\dot{U}(\tilde{P}_i)-\#M(\tilde{P}_i)=\#U(\tilde{P}_i)-1\ . \tag{1}$$

On the other hand, we see the following.

$$\#\dot{U}(P)-\#M(P)=(\#\dot{U}(\tilde{P}_1)+\#\dot{U}(\tilde{P}_2))-(\#M(\tilde{P}_1)+\#M(\tilde{P}_2)-1)\ . \tag{2}$$

Combining (1) and (2), we have the required result immediately.

Theorem 2. *Let P be a closed fake surface with $H_1(P)=0$. Suppose $u=\#U(P)\neq 0$ and $+B(P)\neq\emptyset$. Then, we obtain*

$$\#(+B(P))=2^{u-1}\ .$$

Proof. We can prove Theorem 2 by induction on u again. When $u=1$, we see the conclusion by Lemma 4. And, hence, we may assume $u\geqq 2$. In this case, remember Case 2 of the proof of Theorem 1. We use the same notations and it is sufficient to show that we obtain exactly two elements α and β of $+B(P)$ from η_1 and η_2. Suppose that α and β be obtained by identifying $(\eta_1|b)$ and $(\eta_2|b)$ by an orientation preserving equivalence f and an orientation reversing one g, respectively. Suppose that α and β are equivalent by an equivalence h. Then, by Lemma 5, it may be assumed that $h|(\alpha|U(P))$ is orientation preserving. However, if $h|(\alpha|U(P_1))$ is orientation preserving, then, $h|(\alpha|U(P_2))$ has to be orientation reversing, because f is orientation preserving but g is not. This gives a contradiction. Thus, α and β are not equivalent. Hence, we obtain $\#(+B(P))=2^{u-1}$.

REFERENCES

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Errata

A CORRECTION TO A THEOREM OF MINE

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The Theorem 12 of the paper entitled "On Conharmonically Recurrent Spaces of Second Order" published in the Yokohama Mathematical Journal, Vol. XXI, No. 2, 1973, will read

'If a ${}^2L_{2n}$ $(n>1)$ be a product space $V_n \times V_n$, then each of the decomposition spaces is an Einstein Space.'