ELEMENTARY SURGERY ON SEIFERT FIBER SPACES

By

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Let F be an orientable Seifert fiber space. The structure of all 3-manifolds which are obtained by removing disjoint fibered solid tori in F and sewing them back differently is described.

Since the complement of a fibered solid torus in F is a Seifert fiber space it suffices to investigate the class of all manifolds that are a sum of a Seifert fiber space M and solid tori. Here a *sum* of M and solid tori $V_1, \cdots, V_m$ is the manifold obtained from M and $V_1, \cdots, V_m$ by identifying a component T_i of ∂M with ∂V_i under a homeomorphism $f_i : \partial V_i \to T_i$ $(i=1, \cdots, m)$. The case that M is a Seifert fiber space with orbit surface a disk has been studied in [1]. In particular, the case that F is the complement of a regular neighborhood in S^3 of a torus knot, has been considered in detail in [2]. A related result about graph manifolds (a generalization of Seifert fiber spaces) has been obtained in [4, Satz 6.3.3].

The *connected sum* $M_1 \# M_2$ of two 3-manifolds is the manifold obtained by removing 3-balls in $\operatorname{int}(M_i)$ and identifying the resulting 2-sphere boundaries (under an orientation reversing homeomorphism). If M is a 3-manifold we denote by $\hat{M}$ the manifold obtained from M by capping off each 2-sphere of ∂M with a 3-cell.

If F is a Seifert fiber space [3] we denote by $p : F \to f$ the projection onto the orbit surface f. The image of an exceptional fiber is an exceptional point of f. Note that a Seifert fiber space without exceptional fiber is a S^1-bundle over f.

T^3 denotes the solid torus $D^2 \times S^1$, and

$$(S^1 \times S^2)^n = (S^1 \times S^2) \# \cdots \# (S^1 \times S^2) \qquad (n \geq 0 \quad \text{copies}),$$

where $(S^1 \times S^2)^0 = S^3$; similarly

$$(T^3)^n = T^3 \# \cdots \# T^3 \qquad (n \geq 0 \quad \text{copies}),$$

where $(T^3)^0 = S^3$.

By a lens space we mean the sum of two solid tori different from $S^1 \times S^2$. A lens space is trivial if it is S^3.

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Proposition 1. *Fet F be an orientable S^1-bundle over an orientable surface f of genus g and $n\geq 1$ boundary components. Suppose M is a sum of F and n solid tori $V_1, \cdots, V_n$ such that the meridian of each V_i is homologous (on ∂V_i) to a fiber of F. Then $M\approx(S^1\times S^2)^{2g+n-1}$.*

Proof. By a small deformation of the fibering of F we can assume that the meridian of V_i is a fiber $(i=1, \cdots, n)$.

(a) Assume $g=0$. If $n=1$, then $M\approx S^3\approx(S^1\times S^2)^0$. Thus assume f has $n>1$ boundary components $r_1, \cdots, r_n$. Let l be a simple arc on f from r_1 to r_2, $\partial l=p_1\cup p_2=l\cap\partial f$, and U a regular neighborhood of l on f such that $U\cap r_i=a_i$, an arc $(i=1, 2)$, and $\partial U-(a_1\cup a_2)$ consists of two arcs l_1, l_2. Let $f'=cl(f-U)$ and $F'=p^{-1}(f')$. Now $p^{-1}(p_i)$ bounds a disk D_i in V_i $(i=1, 2)$. Let $D_i\times I$ be a regular neighborhood of D_i in V_i such that $D_i=D_i\times 1/2$, $p^{-1}(\partial a_i)=\partial D_i\times 0\cup\partial D_i\times 1$, and let B_i be the 3-ball $cl(V_i-D_i\times I)$. Let $M'=F'\cup V_3\cup\cdots\cup V_n\cup B_1\cup B_2$ where the 3-ball B_i is attached along the annulus $p^{-1}(cl(r_i-a_i))$. M is obtained from M' by identifying the two 2-spheres $p^{-1}(l_i)\cup D_i\times 0\cup D_j\times 1$ of $\partial M'$. Hence $M\approx\hat{M}'\# S^1\times S^2$. Here $\hat{M}'$ is obtained from M' by extending the attaching maps of $\partial B_i\to\partial F'$ to an attaching map of $\partial V\to\partial F'$, where V is a solid torus. Hence $\hat{M}'\approx F'\cup V\cup V_3\cup\cdots\cup V_n$. By induction, since f' has $n-1$ boundary components, $\hat{M}'\approx(S^1\times S^2)^{n-2}$.

(b) Assume $g\geq 1$. Let l be a simple nonseparating arc on f such that $\partial l=r_1\cap l=p_1\cup p_2$. Let U be a regular neighborhood of l on f with $U\cap\partial f=a_1\cup a_2$, two disjoint arcs on r_1, and let $f'=cl(f-U)$. Let l_1, l_2 be the components of $cl(\partial U-a_1\cup a_2)$ and b_1, b_2 the components of $cl(r_1-a_1\cup a_2)$. Now $p^{-1}(p_i)$ bounds a disk D_i in V_1 $(i=1, 2)$. Let $D_i\times I$ be a regular neighborhood of D_i in V_1 with $p^{-1}(\partial a_i)=\partial D_i\times 0\cup\partial D_i\times 1$, and let B_1, B_2 be the 3-balls of $cl(V_1-D_1\times I\cup D_2\times I)$. Let $M'=F'\cup V_2\cup\cdots\cup V_n\cup B_1\cup B_2$, where $F'=p^{-1}(f')$ and where B_i is attached along $p^{-1}(b_i)$ $(i=1, 2)$. As before, $M\approx\hat{M}'\# S^1\times S^2$, with $\hat{M}'=F'\cup V_1'\cup V_1''\cup V_2\cup\cdots\cup V_n$, where $\partial V_1'$ and $\partial V_1''$ is attached to $p^{-1}(l_1\cup b_1)$ and $p^{-1}(l_2\cup b_2)$, resp., such that a meridian of V_1' and V_1'' is a fiber of F'. Since genus $(f')=g-1$ and $\partial f'$ has $n+1$ components, it follows by induction on g that $\hat{M}'\approx(S^1\times S^2)^{2(g-1)+(n+1)-1}$ and hence $M\approx(S^1\times S^2)^{2g+n-1}$.

Proposition 2. *Let F be an orientable Seifert fiber space with $r\geq 0$ exceptional fibers and with orientable orbit surface f of genus $g\geq 0$ and $n\geq 1$ boundary components. Let M be the sum of F and n solid tori $V_1, \cdots, V_n$.*

(a) If a meridian of each V_i is not homologous to a fiber of F then M is an orientable Seifert fiber space (with at most $r+n$ exceptional fibers and with a closed orientable orbit surface of genus g).

(b) *If a meridian of each* V_i *is homologous to a fiber of* F *then* $M \approx L_1 \# \cdots \# L_r \# (S^1 \times S^2)^{2g+n-1}$ *where* L_i *is a nontrivial lensspace* $(i=1, \cdots, r)$.

(c) *If a meridian of* V_i *is not homologous to a fiber for* $1 \leq i \leq q$, *but a meridian of* V_j *is homologous to a fiber for* $q < j \leq n$, *then* $M \approx L_1 \# \cdots \# L_m \# (S^1 \times S^2)^{2g+n-q-1}$ *for some* m *with* $r \leq m \leq r+q$.

Proof. (a) In this case the fibering of F can be extended to a Seifert fibering of M.

(b) Let D be a disk on f such that int D contains the r exceptional points of f, $D \cap \partial f = \partial D \cap \partial f = a$ is an arc on $r_1 = p(\partial V_1)$. As in the proof of proposition 1, let $D_1 \times I$ be a 3-ball in V_1 determined by the meridianal annulus $p^{-1}(a)$ in V_1 and $B_1 = cl(V_1 - D_1 \times I)$. Then $M \approx \hat{M}_1 \# \hat{M}_2$, where $M_1 = p^{-1}(cl(f-D)) \cup B_1$, $M_2 = p^{-1}(D) \cup D_1 \times I$. By lemma 3 of [1], M_2 is a connected sum of r non-trivial lens spaces minus an open 3-ball, and by proposition 1, $M_1 = (S^1 \times S^2)^{2g+n-1} - $3-ball.

(c) The fibering of F can be extended to a fibering of $F' = F \cup V_1 \cup \cdots \cup V_q$ with at most $r+q$ exceptional fibers. Thus (c) follows by applying (b) to $F' \cup V_{q+1} \cup \cdots \cup V_n$.

We now consider the case that not all boundary components of a Seifert fiber space F are filled in by solid tori.

Lemma. *Let* M *be an orientable 3-manifold such that* ∂M *contains only one 2-sphere component* S. *Let* N *be obtained from* M *by adding a 1-handle at* S. *Then* $N \approx \hat{M} \# T^3$.

Proof. Let V' be obtained from T^3 by removing a 3-ball from int(T^3) and let S' be the 2-sphere component of $\partial V'$. Then $\hat{M} \# T^3 = M \cup V'$, where the union is along S and S'. Let D be a meridianal disk in V', $D \cap \partial V' = \partial D$ and let U be a regular neighborhood of D in V' with $U \cap \partial V'$ a regular neighborhood of ∂D in $\partial V'$. Then $M \cup cl(V' - D) \approx M$, since $cl(V' - D)$ is just a collar of ∂M. Hence $M \cup V'$ is obtained from M by adding a 1-handle at S.

Proposition 3. *Let* F *be a Seifert fiber space with no exceptional fibers and orbitsurface* f *a 2-sphere with* n *holes,* $n \geq 2$. *Let* M *be the sum of* F *and solid tori* $V_1, \cdots, V_m$ $(1 \leq m \leq n)$ *such that the meridian of* V_i *is homologous to a fiber of* F $(i=1, \cdots, m)$. *Then* $M \approx (S^1 \times S^2)^{m-1} \# (T^3)^{n-m}$.

Proof. (a) First assume $m=1$, $n=2$. Let r_1 be the boundary component of f for which $p^{-1}(r_1) = \partial V_1$. Let U be a regular neighborhood on f of an arc l from r_1 to the other boundary r_2, such that $U \cap r_i = a_i$, an arc $(i=1, 2)$. Then the annuli $p^{-1}(a_1)$ and $p^{-1}(cl(r_1 - a_1))$ determine 3-balls B_1 and B_2 in V, respectively.

Now $p^{-1}(U) \cup B_1 \approx$ 3-ball and $p^{-1}(cl(f-U)) \cup B_2 \approx$ 3-ball, and M is a union of two 3-balls along two disjoint disks in their boundaries, hence $M \approx T^3$.

(b) Assume $m=1$, $n>2$. Let U be as above, then $M=M_1 \cup B$, where B is a 3-ball and where the union is along two disjoint disks in ∂M_1 and ∂B, and where $M_1=p^{-1}(cl(f-U)) \cup B_2$. By induction on n, $M_1 \approx (T^3)^{(n-2)}-$3-ball and M is obtained from M_1 by adding a 1-handle to the boundary sphere of M_1. By the lemma, $M \approx \hat{M}_1 \# T^3 \approx (T^3)^{n-1}$.

(c) $1 \leq m \leq n$. Write $M \approx F \cup V_1 \cup \cdots \cup V_m$. By (b) $F \cup V_1 = (T^3)^{n-1}$. For each summand T^3 we have a fibering on ∂T^3 (induced from the fibering of F) such that the fiber of ∂T^3 is a meridian of T^3. Hence $V_2, \cdots, V_m$ are attached to $F \cup V_1$ such that the meridian of V_i is identified with the meridian of one of the T^3, i.e. $T^3 \cup V_i \approx S^1 \times S^2$. Hence $M \approx (S^1 \times S^2)^{m-1} \# (T^3)^{n-m}$.

Proposition 4. *Let F be an orientable Seifert fiber space without exceptional fibers and orientable orbit surface of genus g and n boundary components. Suppose M is a sum of F and m solid tori $V_1, \cdots, V_m$ $(1 \leq m \leq n)$ such that the meridian of V_i is homologous to a fiber of F $(i=1, \cdots, m)$. Then $T \approx (S^1 \times S^2)^{2g+m-1} \# (T^3)^{n-m}$.*

Proof. For $g=0$ this is proposition 3. Suppose $g>0$. Let r_i be the boundary component of f with $p^{-1}(r_i)=\partial V_i$. Let U be a regular neighborhood of a non-separating arc l on f with $\partial l = l \cap \partial f = l \cap r_1$, such that $U \cap r_1$ consists of two arcs a_1, a_2. Let $D_i \times I$ be the 3-ball in V_1 determined by $p^{-1}(a_i)$ $(i=1,2)$ and let B_1, B_2 be the components of $cl(V_1 - D_1 \times I \cup D_2 \times I)$. Then $M=M_1 \cup M_2$, where $M_1 = p^{-1}(cl(f-U)) \cup B_1 \cup B_2 \cup V_2 \cup \cdots \cup V_r$ and $M_2 = p^{-1}(U) \cup D_1 \times I \cup D_2 \times I \approx S^2 \times I$, and where the union is over $S^2 \times 0$ and $S^2 \times 1$. Extending the attaching maps of $\partial B_i \to \partial M_1$ to an attaching map $\partial T_i^3 \to \partial M_1$ $(i=1, 2)$, we see that $\hat{M}_1 \approx p^{-1}(f') \cup T_1^3 \cup T_2^3 \cup V_2 \cup \cdots \cup V_m$, where $f'=cl(f-U)$, genus $(f')=g-1$, and $\partial f'$ consists of $n+1$ components. By induction on g,

$$\hat{M}_1 \approx (S^1 \times S^2)^{2(g-1)+(m+1)-1} \# (T^3)^{(n+1)-(m+1)},$$

and hence

$$M \approx \hat{M}_1 \# (S^1 \times S^2) \approx (S^1 \times S^2)^{2g+m-1} \# (T^3)^{n-m}.$$

Proposition 5. *Let F be an orientable Seifert fiber space with r exceptional fibers and with orientable orbit surface f of genus g and n boundary components. Let M be a sum of F and m solid tori $V_1, \cdots, V_m$ $(1 \leq m \leq n)$. Then*

(a) *If the meridian of V_i is not homologous to a fiber of F for $i=1, \cdots, m$, M is a Seifert fiber space with at most $r+m$ exceptional fibers.*

(b) *If the meridian of* V_i *is homologous to a fiber of* F *for* $1 \leq i \leq q$, *but not homologous to a fiber for* $q < i \leq m$, *then* $M \approx L_1 \# \circ\circ\circ \# L_{m-q+r} \# (S^1 \times S^2)^{2g+q-1} \# (T^3)^{n-q}$ *where* L_i *are lensspaces (at least* r *of which are non trivial).*

Proof. This follows from the previous propositions as in the proof of proposition 2.

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