POLYNOMIAL COEFFICIENTS OF ENTIRE SERIES

By

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1. The growth of an entire function

$$f(z)=\sum_{n=0}^{\infty} a_n z^n \,, \tag{1.1}$$

is studied with the help of growth constants ρ, λ and T, known as order, lower order and type respectively and defined as follows:

$$\begin{matrix}\rho \\ \lambda\end{matrix} = \lim_{r\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log\log M(r)}{\log r} \,,$$

and

$$T=\limsup_{r\to\infty} \frac{\log M(r)}{r^{\rho}} \,, \qquad (0<\rho<\infty) \,,$$

where $M(r)=\max_{|z|=r} |f(z)|$. The coefficient equivalents of order ρ and type T of an entire function given by (1.1) are known [1, pp. 9-11]. Thus,

$$\rho=\limsup_{n\to\infty} \frac{n\log n}{\log |a_n|^{-1}} \,, \tag{1.2}$$

and

$$e\rho T=\limsup_{n\to\infty} n|a_n|^{\rho/n} \,. \tag{1.3}$$

A coefficient formula analogous to (1.2) does not hold always for the lower order. *Juneja* [3] and *Juneja* and *Kapoor* [4] obtained formulae for the lower order involving coefficients which hold for every entire function. Thus, if $f(z)$, given by (1.1), be an entire function of lower order λ, then

$$\lambda=\max_{\{m_k\}}\left[\liminf_{k\to\infty} \frac{m_k \log m_{k-1}}{\log |a_{m_k}|^{-1}}\right]. \tag{1.4}$$

$$=\max_{\{m_k\}}\left[\liminf_{k\to\infty} \frac{(m_k-m_{k-1})\log m_{k-1}}{\log |a_{m_{k-1}}/a_{m_k}|}\right]. \tag{1.5}$$

However for entire functions of infinite order the growth constants defined above do not give any satisfactory information about their growth. *Sato* [6] studied the growth of such functions by introducing the concept of 'index' of an entire function. Thus, if for an entire function $f(z)$,

(1.6) $$\rho(q)=\limsup_{r\to\infty}\frac{\log^{[q]} M(r)}{\log r},$$

and

(1.7) $$\kappa(q)=\limsup_{r\to\infty}\frac{\log^{[q-1]} M(r)}{r^{\rho(q)}}, \qquad (0<\rho(q)<\infty),$$

where $\log^{[0]} M(r)=M(r)$ and $\log^{[q]} M(r)=\log(\log^{[q-1]} M(r))$, $q=1, 2, 3, \cdots$, then $f(z)$ is said to be of index q if $\rho(q-1)=\infty$ and $\rho(q)<\infty$. We call $\rho(q)$ as the q-order and $\kappa(q)$ as q-type of the entire function $f(z)$ having the index q. The coefficient equivalents of $\rho(q)$ and $\kappa(q)$, as obtained by *Sato*, for the entire function given by (1.1) and having index q are as following:

(1.8) $$\rho(q)=\limsup_{n\to\infty}\frac{n\log^{[q-1]} n}{\log |a_n|^{-1}},$$

and

(1.9) $$\kappa(q)=\limsup_{n\to\infty}\left(\log^{[q-2]}\frac{n}{e\rho(q)}\right)|a_n|^{\rho(q)/n}.$$

Analogous to the concept of lower order, the lower q-order $\lambda(q)$ of an entire function $f(z)$ can be defined as

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q]} M(r)}{\log r}.$$

Recently, *Rice* [5] has extended the results (1.2) and (1.3) by considering the polynomial expansion of $f(z)$ of the form

(1.10) $$f(z)=\sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1},$$

where $p(z)$ is a polynomial of degree ζ and $q_k(z)$ is a uniquely determined polynomial of degree $\zeta-1$ or less. If Γ_R be the lemniscate $\Gamma_R=\{z: |p(z)|=R\}$, $||\Gamma_R||$ be the length of Γ_R and $M(\Gamma_R)=||f(z)||_{\Gamma_R}=\max_{z\in\Gamma_R}|f(z)|$, then using the estimate

(1.11) $$||\Gamma_R||=2\pi R^{1/\zeta}(1+o(1)) \quad \text{as} \quad R\to\infty,$$

he showed that $f(z)$ given by (1.10) is an entire function of order ρ, if and only if,

(1.12) $$\limsup_{R\to\infty}\frac{\log\log M(\Gamma_R)}{\log R}=\rho/\zeta,$$

and that $f(z)$ is of order $\rho>0$ and type $T(0<T<\infty)$, if and only if,

(1.13)
$$\limsup_{R\to\infty} \frac{\log M(\Gamma_R)}{R^{\rho/\zeta}} = T .$$

His generalizations of (1.2) and (1.3) read as follows:

Let α† be fixed. Then $f(z)$, given by (1.10), is an entire function of order $\rho>0$, if and only if,

(1.14)
$$\rho = \zeta \limsup_{n\to\infty} \frac{n \log n}{\log \| q_n(z) \|_{\Gamma_\alpha}^{-1}} ,$$

and that it is of order $\rho>0$ and type $T(0<T<\infty)$, if and only if,

(1.15)
$$e\rho T = \zeta \limsup_{n\to\infty} n(\| q_n(z) \|_{\Gamma_\alpha})^{\rho/\zeta n} .$$

(1.14) and (1.15) depict the influence of the rate of decrease of $\| q_n(z) \|_{\Gamma_\alpha}$ on the growth of $f(z)$. But, as in the case of power series, these results also do not give any precise information about the growth of the function $f(z)$ if it is of infinite order. For this purpose, in the present paper, we obtain formulae for the q-order, lower q-order and q-type involving polynomial coefficients of an entire function $f(z)$ given by (1.10). Our results include the results of *Sato* [6], *Rice* [5], *Juneja* [3] and *Juneja* and *Kapoor* [4]. For $q=2$, the method adopted in proving Theorems 1 and 2 yields a short alternative method to obtain (1.14) and (1.15).

To avoid trivial cases, we shall assume throughout that $f(z)$ is an entire transcendental function.

2. In this section we obtain results analogous to (1.14) and (1.15) for the q-order and q-type. We require the following two lemmas:

Lemma 1. *$f(z)$, given by* (1.10), *is an entire function of q-order $\rho(q)$ and lower q-order $\lambda(q)$, if and only if,*

(2.1)
$$\lim_{R\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log^{[q]} M(\Gamma_R)}{\log R} = \begin{matrix}\rho(q)/\zeta\\ \lambda(q)/\zeta\end{matrix} .$$

Further, if the q-order of $f(z)$ is $\rho(q)$ (>0), then it is of q-type $\kappa(q)$, if and only if,

(2.2)
$$\limsup_{R\to\infty} \frac{\log^{[q-1]} M(\Gamma_R)}{R^{\rho(q)/\zeta}} = \kappa(q) ,$$

ζ being the degree of $p(z)$.

† Throughout our discussions in this paper α will denote a fixed constant not less than 1.

The lemma follows on the lines similar to those of Rice [5, Lemma 3]. Hence we omit its proof.

Lemma 2. *Let* α *be fixed and* $f(z)=\sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}$ *be an entire function having q-order* $\rho(q)$, *lower q-order* $\lambda(q)$ *and q-type* $\kappa(q)$, *then*

$$\begin{matrix}\rho(q)/\zeta \\ \lambda(q)/\zeta\end{matrix} = \lim_{R\to\infty} \begin{matrix}\sup \\ \inf\end{matrix} \frac{\log^{[q]} H_\alpha(R)}{\log R}, \tag{2.3}$$

and

$$\kappa(q) = \limsup_{R\to\infty} \frac{\log^{[q-1]} H_\alpha(R)}{R^{\rho(q)/\zeta}}, \qquad (\rho(q)>0), \tag{2.4}$$

where $H_\alpha(R)=\sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} R^k$ *and* ζ *is the degree of* $p(z)$.

Proof. Let $R>\alpha$. Then since [7, p. 77]

$$\|q_k(z)\|_{\Gamma_R} \leq \|q_k(z)\|_{\Gamma_\alpha} R^{\zeta-1},$$

we have, for $z\in\Gamma_R$

$$|f(z)| \leq \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R} \|p(z)\|_{\Gamma_R}^{k-1},$$

$$\begin{aligned} M(\Gamma_R) &\leq \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_R} \|p(z)\|_{\Gamma_R}^{k-1} \\ &\leq R^{\zeta-2} \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} R^k \\ &= R^{\zeta-2} H_\alpha(R). \end{aligned} \tag{2.5}$$

It is known [5, Lemma 2] that if $f(z)$, given by (1.10), is analytic in Γ_R, then there exists a polynomial $Q(z)$ of degree $\zeta-1$ independent of k and R such that for $\alpha<R$ and $k=1, 2, \cdots$

$$\|q_k(z)\|_{\Gamma_\alpha} \leq \frac{\|\Gamma_R\| M(\Gamma_R)}{2\pi R^k} \|Q(z)\|_{\Gamma_R}. \tag{2.6}$$

Hence, in view of (1.11), we have for every $\eta>0$,

$$\begin{aligned} H_\alpha(R) &= \sum_{k=1}^{\infty} \|q_k(z)\|_{\Gamma_\alpha} R^k \\ &\leq M(\Gamma_{R+\eta}) \frac{\|\Gamma_{R+\eta}\| \, \|Q\|_{\Gamma_{R+\eta}}}{2\pi} \sum_{k=1}^{\infty} \left(\frac{R}{R+\eta}\right)^k \\ &\leq M(\Gamma_{R+\eta}) \frac{(R+\eta)^{1/\zeta}(1+o(1))\|Q\|_{\Gamma_\alpha} R^\zeta}{\eta} \\ &= M(\Gamma_{R+\eta}) \frac{R^{\zeta+1/\zeta}(1+o(1))}{\eta} \|Q\|_{\Gamma_\alpha}. \end{aligned} \tag{2.7}$$

Comparing (2.5) and (2.7), we get for all $R>\alpha$ and $\eta>0$,

$$M(\Gamma_R)\leq R^{\zeta-2}H_\alpha(R)\leq M(\Gamma_{R+\eta})\frac{R^{2\zeta+1/\zeta-2}(1+o(1))||Q||_{\Gamma_\alpha}}{2\pi\eta}. \tag{2.8}$$

Now using Lemma 1, (2.3) and (2.4) follow from (2.8). Hence the lemma is proved.

Theorem 1. *Let α be fixed and $f(z)=\sum_{k=1}^{\infty}q_k(z)[p(z)]^{k-1}$ be an entire function having q-order $\rho(q)$, then*

$$\rho(q)/\zeta=\limsup_{k\to\infty}\frac{k\log^{[q-1]}k}{\log||q_k(z)||_{\Gamma_\alpha}^{-1}}, \tag{2.9}$$

where ζ is the degree of $p(z)$.

Proof. Consider the entire function

$$H_\alpha(w)=\sum_{k=1}^{\infty}||q_k(z)||_{\Gamma_\alpha}w^k.$$

It is easily seen that $H_\alpha(w)$ is of index q and if its q-order be $\rho^*(q)$, then, by Lemma 2, $\rho(q)=\zeta\rho^*(q)$. Now applying (1.8) to $H_\alpha(w)$, we get

$$\rho^*(q)=\limsup_{k\to\infty}\frac{k\log^{[q-1]}k}{\log||q_k(z)||_{\Gamma_\alpha}^{-1}},$$

and hence (2.9) follows.

Proceeding on the similar lines the following theorem can be easily obtained.

Theorem 2. *Let α be fixed and $f(z)=\sum_{k=1}^{\infty}q_k(z)[p(z)]^{k-1}$ be an entire function of q-order $\rho(q)$ (>0) and q-type $\kappa(q)$. Then,*

$$\kappa(q)=\limsup_{k\to\infty}\left\{\log^{[q-2]}\left(\frac{k}{e\rho(q)}\right)\right\}(||q_k(z)||_{\Gamma_\alpha})^{\rho/k\zeta},$$

where ζ is the degree of $p(z)$.

Remark. For $q=2$, the above theorems include the results of *Rice* [5], which were obtained by a different technique.

3. In this section we obtain formulae involving polynomial coefficients for the lower q-order of an entire function given by (1.10). We require the following lemmas:

Lemma 3. *Let $f(z)$, given by (1.1) be an entire function of index q and lower q-order $\lambda(q)$ and let $\mu(r)$ and $\nu(r)$ denote respectively the maximum term* *and central index of* $f(z)$ *for* $|z|=r$, *i. e.*, $\mu(r)=\max_{n\geq 0}\{|a_n|r^n\}$ *and* $\nu(r)=\max\{n:\mu(r)=|a_n|r^n\}$. *Then,*

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q-1]}\nu(r)}{\log r}=\liminf_{r\to\infty}\frac{\log^{[q]}\mu(r)}{\log r}\ . \tag{3.1}$$

The lemma follows easily on the same lines as those of *Whittaker* [8] for $q=2$, so we omit the proof.

Lemma 4. *Let* $f(z)$, *given by* (1.1), *be an entire function of index* q *and lower* q*-order* $\lambda(q)$ *and let* $\{n_k\}$ *denote the range of the step function* $\nu(r)$, *then*

$$\lambda(q)=\liminf_{k\to\infty}\frac{\log^{[q-1]}n_k}{\log\rho(n_{k+1})}\ , \tag{3.2}$$

where $\rho(n_k)$*'s denote the jump points of* $\nu(r)$.

Proof. It is clear that

$$\nu(r)=n_k\quad\text{when}\quad\rho(n_k)\leq r<\rho(n_{k+1})\ ,$$

and that

$$\rho(n_k)<\rho(n_k+1)=\cdots=\rho(n_{k+1})\ .$$

$$\lambda(q)=\liminf_{r\to\infty}\frac{\log^{[q-1]}\nu(r)}{\log r}\geq\liminf_{k\to\infty}\frac{\log^{[q-1]}(n_k+1)}{\log\rho(n_{k+1})}$$
$$=\liminf_{k\to\infty}\frac{\log^{[q-1]}(n_k+1)}{\log\rho(n_k+1)}\geq\lambda(q)\ ,$$

which gives (3.2).

Remark. For $q=2$, the relation (3.2) is due to *Gray* and *Shah* [2, Lemma 1].

Lemma 5. *Let* $f(z)=\sum_{k=0}^{\infty}a_k z^{n_k}$ *be an entire function of index* q *and lower* q*-order* $\lambda(q)$ *such that* $\psi(k)\equiv|a_k/a_{k+1}|^{1/(n_{k+1}-n_k)}$ *forms an increasing function of* k *for* $k>k_0$, *then*

$$\lambda(q)=\liminf_{k\to\infty}\frac{(n_{k+1}-n_k)\log^{[q-1]}n_k}{\log|a_k/a_{k+1}|}\ . \tag{3.3}$$

Proof. For $q=2$, this result is due to *Juneja* and *Kapoor* [4]. We note that since $\psi(k)$ forms an increasing function of k for $k>k_0$, we have

$$\nu(r)=n_k\quad\text{for}\quad\psi(k-1)\leq r<\psi(k)\ ,$$

so that for sufficiently large k, $\rho(n_k)=\psi(k-1)$, $\rho(n_{k+1})=\psi(k)$. Substituting the value of $\rho(n_{k+1})$ in (3.2) we get (3.3).

Lemma 6. *Let* $\{n_k\}$ *be an increasing sequence of positive integers and* $\{a_n\}$ *be a sequence of complex numbers such that* $|a_{n_k}|<1$ *for* $k>k_0$, *then for* $q\geq 2$,

$$\liminf_{k\to\infty}\frac{n_k \log^{[q-1]} n_{k-1}}{\log|a_{n_k}|^{-1}}\geq \liminf_{k\to\infty}\frac{(n_k-n_{k-1})\log^{[q-1]} n_{k-1}}{\log|a_{n_{k-1}}/a_{n_k}|}\,. \tag{3.4}$$

The lemma follows exactly on the same lines as those of *Juneja* [3, Lemma 2] for $q=2$, so we omit the proof.

Theorem 3. *Let* α *be fixed and* $f(z)=\sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}$ *be an entire function of index* q *and lower* q*-order* $\lambda(q)$. *Then,*

$$\lambda(q)/\zeta=\max_{\{m_k\}}\left[\liminf_{k\to\infty}\frac{m_k\log^{[q-1]} m_{k-1}}{\log\|q_{m_k}(z)\|_{\Gamma_\alpha}^{-1}}\right], \tag{3.5}$$

where ζ *is the degree of* $p(z)$. *Maximum in* (3.5) *is taken over all increasing sequences* $\{m_k\}$ *of natural numbers.*

Proof. For any arbitrary sequence $\{m_k\}$ of natural numbers, let

$$\liminf_{k\to\infty}\frac{m_k\log^{[q-1]} m_{k-1}}{\log\|q_{m_k}(z)\|_{\Gamma_\alpha}^{-1}}=\theta(\{m_k\})\equiv\theta\,.$$

Since $f(z)$ is an entire function, $0\leq\theta\leq\infty$. First let $0<\theta<\infty$. For any ε such that $\theta>\varepsilon>0$ and for $k>k_0$, we have

$$\|q_{m_k}(z)\|_{\Gamma_\alpha}>(\log^{[q-2]} m_{k-1})^{-m_k/(\theta-\varepsilon)}\,. \tag{3.6}$$

Set,

$$R_k=e(\log^{[q-2]} m_{k-1})^{1/(\theta-\varepsilon)}\,,$$

and let $R_k\leq R\leq R_{k+1}$. Using (2.6) and (3.6), we get for all $k>k_0$,

$$\log M(\Gamma_R)>m_k\log R_k-\frac{m_k}{(\theta-\varepsilon)}\log^{[q-1]} m_{k+1}-\log\left\{\frac{\|\Gamma_R\|}{2\pi}\|Q\|_{\Gamma_R}\right\}.$$

Since [7, p. 77],

$$\|Q\|_{\Gamma_R}\leq\|Q\|_{\Gamma_\alpha}R^{\zeta-1}\,,$$

we have for all $k>k_0$ and for all $R>R'$=sufficient large,

$$\begin{aligned}\log M(\Gamma_R)&>m_k\log R_k-\frac{m_k}{(\theta-\varepsilon)}\log^{[q-1]} m_{k-1}-\left(\frac{1}{\zeta}+\zeta-1\right)\log R+O(1)\\ &=m_k-\left(\zeta+\frac{1}{\zeta}-1\right)\log R+O(1)\\ &\geq\exp^{[q-2]}\left(\frac{R}{e}\right)^{\delta-1}-\left(\frac{1}{\zeta}+\zeta-1\right)\log R+O(1)\,.\end{aligned}$$

Thus, for all $R>R'$,

$$\frac{\log^{[q]} M(\Gamma_R)}{\log (R/e)} \geq \theta-\varepsilon+o(1) ,$$

which on proceeding to limits and using (2.1) gives $\lambda(q)\geq\zeta\theta$. Since the sequence $\{m_k\}$ is arbitrary, we have

$$\lambda(q)/\zeta \geq \max_{\{m_k\}}\left[\liminf_{k\to\infty} \frac{m_k \log^{[q-1]} m_{k-1}}{\log||q_{m_k}(z)||_{\Gamma_\alpha}^{-1}}\right] \equiv \beta \text{ (say)} . \tag{3.7}$$

Now consider the entire function $H_\alpha(w)=\sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} w_k$, where $w=p(z)$. Let $\{n_k\}$ denote the range of the central index of $H_\alpha(w)$. Set, $G_\alpha(w)=\sum_{k=1}^{\infty} ||q_{n_k}(z)||_{\Gamma_\alpha} w^{n_k}$. Thus $G_\alpha(w)$ is an entire function and $G_\alpha(w)$ and $H_\alpha(w)$ have the same maximum term for every w. Hence by Lemma 3, it follows that both have the same lower q-order. Let it be denoted by $\lambda_0(q)$. Since $G_\alpha(w)$ satisfies the hypotheses of Lemma 5, we have

$$\begin{aligned}\lambda_0(q) &= \liminf_{k\to\infty} \frac{(n_k-n_{k-1}) \log^{[q-1]} n_{k-1}}{\log (||q_{nn_{k-1}}(z)||_{\Gamma_\alpha}/||q_n(z)||_{\Gamma_\alpha})} \\ &\leq \liminf_{k\to\infty} \frac{n_k \log^{[q-1]} n_{k-1}}{\log||q_{n_k}(z)||_{\Gamma_\alpha}^{-1}} \qquad \text{(by Lemma 6)} \\ &\leq \max_{\{m_k\}}\left[\liminf_{k\to\infty} \frac{m_k \log^{[q-1]} m_{k-1}}{\log||q_{m_k}(z)||_{\Gamma_\alpha}^{-1}}\right] = \beta .\end{aligned} \tag{3.8}$$

Since, for $\alpha<R$,

$$\begin{aligned} M(\Gamma_R) &\leq \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_R} ||p(z)||_{\Gamma_R}^k \\ &\leq R^{\zeta-1} \sum_{k=1}^{\infty} ||q_k(z)||_{\Gamma_\alpha} R^k \\ &= R^{\zeta-1} H_\alpha(R) ,\end{aligned}$$

by using (3.8), we get

$$M(\Gamma_R) \leq R^{\zeta-1} \exp^{[q-1]} R^{\beta+\varepsilon} ,$$

for a sequence $R_1, R_2, \cdots \to\infty$. Hence, by Lemma 1, we get $\lambda(q)/\zeta\leq\beta$. This when combined with (3.7), proves the theorem completely.

Using Lemmas 5 and 6 the following theorem can be proved similarly.

Theorem 4. *Let α be fixed and $f(z)=\sum_{k=1}^{\infty} q_k(z)[p(z)]^{k-1}(q_k(z)\not\equiv0)$ be an entire function of index q and lower q-order $\lambda(q)$, then*

$$\lambda(q)/\zeta = \max_{\{m_k\}}\left[\liminf_{k\to\infty} \frac{(m_k-m_{k-1}) \log^{[q-1]} m_{k-1}}{\log (||q_{m_{k-1}}(z)||_{\Gamma_\alpha}/||q_{m_k}(z)||_{\Gamma_\alpha})}\right] , \tag{3.9}$$

where ζ is the degree of $p(z)$ and maximum is taken over all increasing sequences of natural numbers.

Remark. Results given in [3] and [4] follow as particular cases of the above theorems by taking $p(z) \equiv z$ and $q=2$.

REFERENCES

[1] Boas, R. P., Jr.: Entire Functions. Academic Press, 1954.
[2] Gray, A. and Shah, S. M.: Holomorphic functions with gap power series. Math. Z. **86** (1965), 375-394.
[3] Juneja, O. P.: On the coefficients of entire series. J. Analyse Math. **24** (1971), 395-401.
[4] Juneja, O. P. and Kapoor, G. P.: On the lower order of entire functions. J. London Math. Soc. **5** (2) (1972), 310-312.
[5] Rice, J. R.: The degree of convergence for entire functions. Duke Math. J. **38** (1971), 429-440.
[6] Sato, D.: On the rate of growth of entire functions of fast growth. Bull. Amer. Math. Soc. **69** (1963), 411-414.
[6] Walsh, J. L.: Interpolation and Approximation Colloquium publications, Vol. 20, Amer. Math. Soc. Providence, R. I., 1960.
[8] Whittaker, J. M.: The lower order of integral functions J. London Math. Soc. **8** (1933), 20-27.

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