PROPERTIES OF C-CONTINUOUS FUNCTIONS

By

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1. Introduction

Professors *Gentry* and *Hoyle* [3] define a function $f: X \to Y$ from one topological space into another to be c-continuous if for each $x \in X$ and each open V containing $f(x)$ and having compact complement, there exists an open U containing x such that $f(U) \subset V$. They then proceed to develop several properties of c-continuous functions. Among the theorems given is the following useful result:

Theorem 1. (*Gentry* and *Hoyle*, [3]) Let $f: X \to Y$ be a function. Then the following are equivalent:

(1) f is c-continuous.

(2) If V is an open subset of Y with compact complement, then $f^{-1}(V)$ is an open subset of X.

These statements are implied by

(3) If C is a compact subset of Y, then $f^{-1}(C)$ is a closed subset of X and, moreover, if Y is Hausdorff, all the statements are equivalent.

In this paper we prove additional results concerning c-continuous functions.

2. Results

We begin by making three rather straightforward observations about c-continuous functions.

Theorem 2. The function $f: X \to Y$ is c-continuous if and only if the inverse image of each closed compact subset of Y is closed in X.

Proof. If the inverse image of each closed compact subset of Y is closed in X, then f is c-continuous by Theorem 1 part (3).

The converse follows from the equivalence of parts (1) and (2) of Theorem 1.

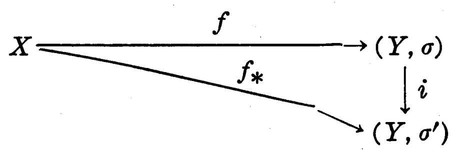
Theorem 3. Let $f: X \to Y$ be c-continuous and injective. If Y is T_1, then X is T_1.

Proof. Theorem 2.

Theorem 4. Let $f: X \to Y$ be c-continuous, closed and surjective. If X is normal and Y is T_1, then Y is also T_2.

Proof. Let $y_1 \neq y_2$ be points in Y. Then $\{y_1\}$ and $\{y_2\}$ are closed compact subsets of Y so that by Theorem 2, $f^{-1}(y_1)$ and $f^{-1}(y_2)$ are closed subsets of X. The normality of X then gives the existence of two disjoint open sets U_1 and U_2 such that $f^{-1}(y_1) \subset U_1$ and $f^{-1}(y_2) \subset U_2$. Since f is closed, there exist open sets V_1 and V_2 of Y such that $y_1 \in V_1$, $f^{-1}(y_1) \subset f^{-1}(V_1) \subset U_1$, $y_2 \in V_2$ and $f^{-1}(y_2) \subset f^{-1}(V_2) \subset U_2$ by Theorem 11.2 [2, p. 86]. Evidently, $V_1 \cap V_2 = \varnothing$ so Y is T_2.

For any topological space (Y, σ), the collection of open sets having compact complements may be used as a base for a topology σ' on Y. The reason is that if U and V are open and have compact complements, then their intersection has a compact complement as may be shown by use of the equality $Y-(U \cap V) = (Y-U) \cup (Y-V)$. Of course $\sigma' \subset \sigma$ and (Y, σ') is always a compact space. With these facts, consider the following commutative diagram where X is any topological space and $f_*(x) = f(x)$ for all $x \in X$:



Evidently, f is c-continuous if and only if f_* is continuous. Also, i is continuous and i^{-1} is c-continuous. We give two further results related to the diagram.

Theorem 5. Let $f: X \to (Y, \sigma)$ be c-continuous. If f_* is closed (open), then f is closed (open).

Proof. Let $M \subset X$ be closed (open). Then $f_*(M)$ is closed (open) and the continuity of i gives $i^{-1}(f_*(M)) = f(M)$ closed (open) in (Y, σ).

We might note that under the hypotheses of Theorem 5, if f is bijective, then f^{-1} is a continuous function.

Theorem 6. Let (Y, σ) be a topological space. If (Y, σ') is Hausdorff, then (Y, σ) is compact and, in particular, $\sigma = \sigma'$.

Proof. Let $\{U_\alpha | \alpha \in \Delta\}$ be an open cover of (Y, σ). Since (Y, σ') is Hausdorff and $\sigma' \subset \sigma$, there exist open sets U and V in (Y, σ) such that $U \cap V = \varnothing$ and both $Y-U$ and $Y-V$ are compact. Thus, from the cover $\{U_\alpha | \alpha \in \Delta\}$ of Y there is a finite subcollection $\{U_{\alpha_{11}}, U_{\alpha_{12}}, \cdots, U_{\alpha_{1n}}\}$ covering $Y-U$ and a finite subcollection $\{U_{\alpha_{21}}, U_{\alpha_{22}}, \cdots, U_{\alpha_{2m}}\}$ convering $Y-V$. The fact that $U \cap V = \varnothing$ then gives the finite subcollection $\{U_{\alpha_{11}}, U_{\alpha_{12}}, \cdots, U_{\alpha_{1n}}, U_{\alpha_{21}}, \cdots, U_{\alpha_{2m}}\}$ as a cover of Y. If follows that (Y, σ) is compact.

Since (Y, σ') is a compact Hausdorff space and $\sigma' \subset \sigma$, σ cannot be a strictly larger topology or else the property of compactness is lost contrary to what has just been shown. Therefore, $\sigma = \sigma'$.

Corollary. Let $f: X \to (Y, \sigma)$ be c-continuous. If (Y, σ') is Hausdorff, then f is continuous.

Proof. The theorem gives (Y, σ) compact so that f is continuous by Theorem 5 [3].

The converse of Theorem 6 does not hold as is shown by any finite space which is not Hausdorff.

We now turn our attention to the interesting relationship between c-continuous functions and functions which have closed graphs.

Theorem 7. Let $f: X \to Y$ be a function with closed graph. Then f is c-continuous.

Proof. Suppose f is not c-continuous at the point $x \in X$. Then there is an open set V containing $f(x)$ and having a compact complement such that no open set in X containing x maps into V under f. Consider the filterbase $\mathscr{U}(x)$ of all open sets in X which contain x. It follows that $\mathscr{U}(x)$ converges to x and that $f(\mathscr{U}(x)) = \{f(U) \mid U \text{ is open and contains } x\}$ is a filterbase in Y. Since $f(U) \cap (Y - V) \neq \varnothing$ for all $U \in \mathscr{U}(x)$, then $\mathscr{B} = \{f(U) \cap (Y - V) \mid U \in \mathscr{U}(x)\}$ is a filterbase and $\mathscr{B}$ is subordinated to $f(\mathscr{U}(x))$. Since $Y - V$ is compact, $\mathscr{B}$ has an accumulation point $y \in Y - V$. From the fact that $f(x) \neq y$, the point $(x, y) \notin G(f)$, the graph of f. Let W be any open set containing (x, y). Then there exist open sets U_1 and V_1 containing x and y, respectively, such that $(x, y) \in U_1 \times V_1 \subset W$. From the fact that $\mathscr{U}(x)$ converges to x, we have the existence of a $U \in \mathscr{U}(x)$ such that $U \subset U_1$. This implies that $f(U) \cap (Y - V) \in \mathscr{B}$; hence, $(f(U) \cap (Y - V)) \cap V_1 \neq \varnothing$ because $\mathscr{B}$ accumulates to y. Therefore, there exists a point $x_1 \in U_1$ such that $f(x_1) \in V_1$ so that $(x_1, f(x_1)) \in U_1 \times V_1 \subset W$ showing $W \cap G(f) \neq \varnothing$. It follows that (x, y) is a cluster point of $G(f)$, but $(x, y) \notin G(f)$. Consequently, $G(f)$ is not closed which contradicts the hypothesis that $G(f)$ is closed. We conclude that f is c-continuous.

Since continuous functions need not have closed graphs, we do not expect that c-continuous functions will have this property either. Then next theorem gives conditions as to when c-continuous functions will have closed graphs.

Theorem 8. Let $f: X \to Y$ be c-continuous and let Y be a locally compact

Hausdorff space. Then f has a closed graph.

Proof. Let (x, y) be a point in $X\times Y$ which does not lie in the graph of f. Then $f(x)\neq y$ and since Y is Hausdorff, there exist open disjoint sets V_1 and V_2 containing $f(x)$ and y, respectively. By Theorem 6.2 (2) [2], there exists an open set V such that $y\in V\subset \bar{V}\subset V_1$ and $\bar{V}$ is compact. By Theorem 1, $f^{-1}(\bar{V})$ is closed in X and does not contain x. Since f is c-continuous, there is an open set U containing x and lying in $X-f^{-1}(\bar{V})$ such that $f(U)\subset Y-\bar{V}$. Therefore, $U\times V$ contains (x, y) but no point of $G(f)$. Thus, the complement of $G(f)$ is open so that $G(f)$ is closed.

If $f: X\to Y$, is a given function, then the function $g: X\to X\times Y$, given by $g(x)=(x, f(x))$, is called the *graph function* with respect to f. There are certain relationships between a function and its graph function, as far as c-continuity is concerned, that we wish to investigate next.

Theorem 9. Let $f: X\to Y$ be a function and X a compact space. If the graph function $g: X\to X\times Y$ is c-continuous, then f is c-continuous.

Proof. Let $x\in X$ and V be an open set containing $f(x)$ such that $Y-V$ is compact. Then $P_Y^{-1}(V)$ is open in $X\times Y$ and, since X and $Y-V$ are compact, $X\times(Y-V)=(X\times Y)-P_Y^{-1}(V)$ is compact. Thus $P_Y^{-1}(V)$ is an open set in $X\times Y$ having a compact complement. Therefore, there exists an open set U containing x such that $g(U)\subset P_Y^{-1}(V)$. It follows that $P_Y g(U)=f(U)\subset V$ so that f is c-continuous.

Theorem 10. Let X and Y be metric spaces where the metric space $X\times Y$ has the property that each closed and bounded subset is compact. Let $f: X\to Y$ be given. If the graph function $g: X\to X\times Y$ is c-continuous, then f is c-continuous.

Proof. Let $x\in X$ and let V be an open set in Y containing $f(x)$ such that $Y-V$ is compact. Then $Y-V$ is closed and bounded, hence, there exists a positive real number a such that $Y-V$ is contained in the basic open set $B(f(x), a)$. Now let $B(x, b)$ be any basic open set containing x. Then $\overline{B(x, b)\times B(f(x), a)}$ is closed and bounded in $X\times Y$ so is compact by hypothesis. Let $V_1=B(f(x), d)$ where $0<d<c$ and $c=\rho(f(x), Y-V)>0$. It follows that $(B(x, b)\times V_1)\cup[(X\times Y)-\overline{B(x, b)\times B(f(x), a)}]=W$ is an open set containing $(x, f(x))$ having a closed bounded, hence compact, complement. Since g is c-continuous, there exists an open $U\subset B(x, b)$ containing x such that $g(U)\subset W$. From the construction of W, $f(U)=P_Y(g(U))\subset V$ showing f is c-continuous at the point x. Thus f is c-continuous.

A function $f: X \to Y$ may have its graph function c-continuous while f is not c-continuous, as the following example shows.

Example 1. Let R represent the reals with the standard topology and let (R, σ) be the reals with the right-ray topology. (Open sets have the form R, $\emptyset$ or $\{x|x>a\}$.) Then $i:(R, \sigma)\to R$ is not c-continuous. To see this, consider the point $i(1)=1$ and any open set V about $i(1)$ such that (1) the complement of V is compact and (2) there exists a point $z \in R$ such that $z>1$ and $z \notin V$. Then evidently no open $U\subset(R, \sigma)$ containing the point 1 has the property that $f(U)\subset V$.

To see that $g:(R, \sigma)\to(R, \sigma)\times R$ is c-continuous, we need only to note that the only open set in $(R, \sigma)\times R$ having compact complement is $(R, \sigma)\times R$. The reason is that if W is open in $(R, \sigma)\times R$ and $(x_0, y_0)\notin W$, then each point in $\{(x, y_0)|x<x_0\}$ does not belong to W so that the non-compactness of the complement of W follows immediately.

Theorem 11. Let X and Y be metric spaces where Y has the property that each closed and bounded subset of Y is compact. If $f: X\to Y$ is c-continuous, then the graph function $g: X\to X\times Y$ is c-continuous.

Proof. Let $x\in X$ and consider the point $(x, f(x))\in X\times Y$. Let W be an open set in $X\times Y$ containing $(x, f(x))$ such that $(X\times Y)-W$ is compact. Thus $(X-Y)-W$ is closed and bounded. Therefore, there exist basic open sets $B(x, a)$ and $B(f(x), b)$ such that $(X\times Y)-W\subset B(x, a)\times B(f(x), b)$. Since $(x, f(x))$ does not belong to the compact set $(X\times Y)-W$, there exist open sets $B(x, a')$ and $B(f(x), b')$ such that $a'\leq a$, $b'\leq b/2$ and $B(x, a')\times B(f(x), b')\subset W$. Now let $V=B(f(x), b')\cup[Y-\overline{B(f(x), b)}]$. Since $\overline{B(f(x), b)}$ is closed and bounded, hence, compact by hypothesis, we have V an open set containing $f(x)$ which has a compact complement. Since f is c-continuous, there exists an open set $U\subset B(x, a')$ such that $f(U)\subset V$. Therefore, $g(U)=\bigcup_{z\in U}(z, f(z))\subset B(x, a')\times V=B(x, a')\times[B(f(x), b')\cup(Y-\overline{B(f(x), b))}]$ $\subset W$ which implies g is c-continuous at x.

We leave as an open question the existence of a function $f: X\to Y$ which is c-continuous but whose graph function is not c-continuous. There are several conditions given in [3] under which c-continuous functions are also continuous. We offer an additional condition in the next theorem.

Theorem 12. Let $f: X\to Y$ be c-continuous and let X be first countable and let Y be countably compact, locally compact and Hausdorff. Then f is continuous.

Proof. Suppose f is not continuous at the point $x\in X$. Then there exists an open set $V\subset Y$ containing $f(x)$ such that every open $U\subset X$ containing x has the property that $f(U)\not\subset V$. Let $U_1\supset U_2\supset\cdots$ be a countable base at x and let $x_n\in U_n$ be a point such that $f(x_n)\notin V$. Then (x_n) converges to x and the sequence $(f(x_n))$ has an accumulation point $y\notin V$ in the countably compact space Y. There exist open sets V_1 and V_2 such that $f(x)\in V_1\subset V$, $y\in V_2$ and $V_1\cap V_2=\varnothing$ in the Hausdorff space Y. Also, there exists an open set $W\subset Y$ such that $y\in W\subset\bar{W}\subset V_2$ and $\bar{W}$ is compact due to the locally compact Hausdorff hypothesis. Thus, $Y-\bar{W}$ is an open set containing $f(x)$ whose complement is compact. But if U is any open set containing x, there exists a $U_n\subset U$ and a point $x_n\in U_n$ such that $f(x_n)\in W$ due to the fact that $(f(x_n))$ accumulates to y. Consequently, $f(U)\not\subset Y-\bar{W}$. This contradicts the hypothesis that f is c-continuous and implies f is continuous.

Theorem 9 of [3] states that if $f:X\to Y$ is continuous and bijective onto the Hausdorff space Y, then $f^{-1}:Y\to X$ is c-continuous. After two definitions, we show that the condition of continuity may be replaced with the weaker condition of almost-continuity.

Definition 1 [5]. A function $f:X\to Y$ is almost continuous if for each $x\in X$ and each open V containing $f(x)$, there exists an open U containing x such that $f(U)\subset\bar{V}^0$. ($\bar{V}^0$ denotes the interior of the closure of V.)

Definition 2 [4]. A space Y is nearly compact if every open cover of Y has a finite subcollection, the interiors of the closures of which cover Y.

Theorem 13. Let $f:X\to Y$ be an almost-continuous bijective function onto the Hausdorff space Y. Then $f^{-1}:Y\to X$ is c-continuous.

Proof. Let $F\subset X$ be compact. Then $f(F)$ is nearly compact by Theorem 3.2 [4]. But since Y is Hausdorff, $f(F)$ is closed by Theorem 2.1 [1]. Now $(f^{-1})^{-1}(F)=f(F)$ is closed so that f^{-1} is c-continuous by Theorem 1 [3].

We note that by use of Theorem 1 [3], it is not difficult to prove that if $f:X\to Y$ is almost-continuous and Y is Hausdorff, then f is also c-continuous. If the Hausdorff condition on Y is removed, then neither function need imply the other.

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