SOME RESULTS ON FIXED POINTS IN UNIFORM SPACES

By

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(Received January 31, 1973)

1. Introduction

Let M be a complete metric space with metric d. The well-known Banach's fixed point theorem is as follows.

Theorem 1.1. Let T be a mapping of M into itself such that

$$d(Tx, Ty) \leqslant \alpha d(x, y) , \qquad \text{for all } x, y \text{ in } M,$$

where $0<\alpha<1$. Then T has a unique fixed point. In [1], Kannan proved the following theorem.

Theorem 1.2. ([1], Th. 1). Let T_1 and T_2 be two mappings of M into itself such that

$$d(T_1x, T_2y) \leqslant \beta[d(x, T_1x)+d(y, T_2y)] ,$$

for all x, y in M, where $0<\beta<1/2$. Then T_1 and T_2 have a unique common fixed point. Recently Srivastava and Gupta gave the following generalisation of Kannan's Theorem.

Theorem 1.3. ([3], Th. 2.1). Let T_1 and T_2 be two mappings of M into itself and p, q two positive integers such that

$$d(T_1^px, T_2^qy) \leqslant \alpha d(x, T_1^px)+\beta d(y, T_2^qy) ,$$

for all x, y in M, where $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. Then T_1 and T_2 have a unique common fixed point. In the present paper we extend these results to a uniform space and prove some other results.

2. Preliminary definitions and results

Let $(X, \mathscr{U})$ be a uniform space. A net $\{x_n ; n \in D, \geqslant\}$ in X is said to converge to an element x in X if for every member U in $\mathscr{U}$, there is an element N in D such that $(x_n, x) \in U$ for all n in D with $n \geqslant N$. A net $\{x_n ; n \in D, \geqslant\}$ is said to be a Cauchy net if for every U in $\mathscr{U}$, there is an element N in D such that $(x_m, x_n) \in U$ for all m, n in D with $m \geqslant N$ and $n \geqslant N$. The space $(X, \mathscr{U})$ is said to be complete if every Cauchy net in X converges to a point in X and sequentially complete if every Cauchy sequence in X converges to a point in X.

For any pseudometric p on X and any $r>0$, we write

$$V_{(p,r)}=\{(x, y)\,;\, x, y \in X \quad \text{and} \quad p(x, y)<r\}\,.$$

From Th. 15 ([2], p. 188) we see that the uniformity $\mathscr{U}$ on X can be generated by the family $\mathscr{F}$ of all pseudometrics on X which are uniformly continuous on $X\times X$. But we have observed that it is not necessary to take all the members of $\mathscr{F}$ to generate the uniformity $\mathscr{U}$. (See Th. 2.1.).

Let $\mathscr{P}$ be a family of pseudometrics on X generating the uniformity $\mathscr{U}$. Denote by $\mathscr{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i,r_i)}$, where $p_i \in \mathscr{P}$ and $r_i>0$, $i=1, 2, \cdots, n$, (the integer n is not fixed). Then clearly $\mathscr{V}$ is a base for the uniformity $\mathscr{U}$.

Let $V\in\mathscr{V}$. Then $V=\bigcap_{i=1}^{n} V_{(p_i,r_i)}$ where $p_i\in\mathscr{P}$ and $r_i>0$, $i=1, 2, \cdots, n$. For each $\alpha>0$, the set $\bigcap_{i=1}^{n} V_{(p_i,\alpha r_i)}$ belongs to $\mathscr{V}$. We denote this set by αV.

Lemma 2.1. If $V\in\mathscr{V}$ and α, β are positive, then

$$\alpha(\beta V)=(\alpha\beta)\, V\,.$$

Lemma 2.2. If $V\in\mathscr{V}$, and α, β are positive, then

$$\alpha V\subset\beta V \quad \text{when} \quad \alpha<\beta\,.$$

Lemma 2.3. Let p by any pseudometric on X and α, β be any two positive numbers. If

$$(x, y) \in \alpha V_{(p,r_1)}\circ\beta V_{(p,r_2)}\,,$$

then

$$p(x, y)<\alpha r_1+\beta r_2\,.$$

Lemma 2.4. If $V\in\mathscr{V}$ and α, β are positive, then

$$\alpha V\circ\beta V\subset(\alpha+\beta)\, V\,.$$

Note 2.1. Let p be any pseudometric on X and α, β, γ and three positive numbers. If

$$(x, y) \in \alpha V_{(p,r_1)}\circ\beta V_{(p,r_2)}\circ\gamma V_{(p,r_3)}\,.$$

then

$$p(x, y)<\alpha r_1+\beta r_2+\gamma r_3\,.$$

Lemma 2.5. Let $x, y \in X$. Then for every V in $\mathscr{V}$ there is a positive number λ such that $(x, y)\in\lambda V$.

The proofs of the Lemmas 2.1-2.5 are simple.

Lemma 2.6. Let V be any member of $\mathscr{V}$. Then there is a pseudometric p on X such that

$$V=V_{(p,1)}\ .$$

Proof. Let x, y be any two points of X. Then by Lemma 2.5, there is a $\lambda>0$ such that $(x, y)\in\lambda V$. Write

$$A_{(x,y)}=\{\lambda\,;\lambda>0 \quad \text{and} \quad (x, y)\in\lambda V\}\ .$$

Now we define $p(x, y)$ by

$$p(x, y)=\text{Inf}\,\{\lambda\,;\lambda\in A_{(x,y)}\}\ .$$

If $x\in X$, then clearly $(x, x)\in\lambda V$ for any $\lambda>0$. This shows that $A_{(x,x)}=\{\lambda\,;\lambda>0\}$. So

$$p(x, x)=\text{Inf}\,A_{(x,x)}=0\ .$$

Again since V is symmetric it follows that $A_{(x,y)}=A_{(y,x)}$. So

$$p(x, y)=p(y, x)\geqslant 0\ .$$

Now let x, y, z be any three points of X. Choose $\varepsilon>0$ arbitrarily. Take $\alpha=p(x, z)+\varepsilon$ and $\beta=p(z, y)+\varepsilon$. Then $\alpha\in A_{(x,z)}$ and $\beta\in A_{(z,y)}$. That $(x, z)\in\alpha V$, and $(z, y)\in\beta V$. This gives that

$$(x, y)\in\beta V\circ\alpha V=\alpha V\circ\beta V\subset(\alpha+\beta)V\ . \qquad \text{[By Lemma 2.4]}$$

Thus $\alpha+\beta\in A_{(x,y)}$. So

$$p(x, y)\leqslant\alpha+\beta=p(x, z)+p(z, y)+2\varepsilon\ .$$

Since $\varepsilon>0$ is arbitrary we get

$$p(x, y)\leqslant p(x, z)+p(z, y)\ .$$

Therefore p is a pseudometric on X.

Let $x, y\in X$ and $p(x, y)<1$. Choose any α with $p(x, y)<\alpha<1$. Then $\alpha\in A_{(x,y)}$ which gives that $(x, y)\in\alpha V\subset V$. [By Lemma 2.2] So

(1) $$V_{(p,1)}\subset V\ .$$

Again, let $(x, y)\in V$. Since $V\in\mathscr{V}$, we can express $V=\bigcap_{i=1}^{n} V_{(p_i,r_i)}$, $p_i\in\mathscr{P}$ and $r_i>0$. Write $\alpha_i=p_i(x, y)$, then $0\leqslant\alpha_i/r_i<1$, $(i=1, 2, \cdots, n)$. Let $\theta=\max\{\alpha_i/r_i\,; i=1, 2, \cdots, n\}$. Then $0\leqslant\theta<1$. Choose any positive α with $\theta<\alpha<1$. We have

$$p_i(x, y)=\alpha_i=\left(\frac{\alpha_i}{r_i}\right)r_i\leqslant\theta r_i<\alpha r_i\,, \qquad (i=1, 2, \cdots, n)\ .$$

So

$$(x, y) \in \bigcap_{i=1}^{n} V_{(p_i, \alpha r_i)} = \alpha V ,$$

and hence $p(x, y) \leqslant \alpha < 1$. Thus

$$V \subset V_{(p,1)} . \tag{2}$$

From (1) and (2) we get

$$V = V_{(p,1)} .$$

Note 2.2. We shall call p the Minkowski's pseudometric of V in analogy with the Minkowski's Functional of a convex and balanced set in a linear topological space.

Theorem 2.1. Every uniformity for the set X can be generated by a family of pseudometrics on X which are uniformly continuous on $X \times X$.

Proof. Let $\mathscr{U}$ be a uniformity for the set X. Let $\mathscr{B}$ be a subfamily of $\mathscr{U}$ such that $\mathscr{B}$ is a base for $\mathscr{U}$, each member of $\mathscr{B}$ is symmetric and no member of $\mathscr{B}$ is equal to $X \times X$. For each V in $\mathscr{B}$, choose a sequence $\{U_n^{(V)}\}_{n=0}^{\infty}$ of symmetric sets in $\mathscr{U}$ with

$$U_{n+1}^{(V)} \circ U_{n+1}^{(V)} \circ U_{n+1}^{(V)} \subset U_n^{(V)}, \quad \text{where} \quad U_0^{(V)} = X \times X, \quad \text{and} \quad U_1^{(V)} = V.$$

By the Metrization Lemma ([2], Ch. 6, §12, p. 185) there is a pseudometric d_V on X such that

$$U_n^{(V)} \subset \{(x, y) ; d_V(x, y) < 2^{-n+2}\} \subset U_{n-1}^{(V)} . \tag{3}$$

Let $\mathscr{P} = \{d_V ; V \in \mathscr{B}\}$. Denote by $\mathscr{V}$ the uniformity for X generated by the family $\mathscr{P}$ of pseudometrics on X. Write

$$W_{(V,r)} = \{(x, y) ; d_V(x, y) < r\} .$$

Let U be any member of $\mathscr{U}$. Then there is a set V in $\mathscr{B}$ with $V \subset U$. From (3) we have

$$W_{(V,1)} \subset U_1^{(V)} = V .$$

So $W_{(V,1)} \subset U$ which gives that $U \in \mathscr{V}$ and hence $\mathscr{U} \subset \mathscr{V}$.

Next let $W \in \mathscr{V}$. Then there are finite number of members $V_1, V_2, \cdots, V_m$ in $\mathscr{B}$ and $r_i > 0$, $(i = 1, 2, \cdots, m)$ such that

$$\bigcap_{i=1}^{m} W_{(V_i, r_i)} \subset W .$$

Choose positive integers $n_1, n_2, \cdots, n_m$ such that

$$2^{-m_i+2} < r_i , \qquad (i = 1, 2, \cdots, m) .$$

From (3) we have

$$U_{n_i}^{(V_i)} \subset \{(x, y)\,;\, d_{V_i}(x, y) < 2^{-n_i+2}\} < W_{(V_i, r_i)}\,, \qquad (i=1, 2, \cdots, m)\,.$$

Write $U = \bigcup_{i=1}^{m} U_{n_i}^{(V_i)}$. Then $U \in \mathscr{U}$ and $U \subset \bigcap_{i=1}^{m} W_{(V_i, r_i)} \subset W$, which gives that $W \in \mathscr{U}$, and so $\mathscr{V} \subset \mathscr{U}$. Therefore we have

$$\mathscr{V} = \mathscr{U}\,.$$

Since each $W_{(V, r)}$ is a member of $\mathscr{U}$, it follows from Th. 11 ([2], Chap. VI, p. 183) that d_V, $(V \in \mathscr{B})$ is uniformly continuous on $X \times X$.

3. Results on fixed point of operators

In this section we assume that $(X, \mathscr{U})$ is a uniform space which is sequentially complete and also a Hausdorff space. Further we suppose that $\mathscr{P}$ is a fixed family of pseudometrics on X which generates the uniformity $\mathscr{U}$. We denote by $\mathscr{V}$ the family of all sets of the form $\bigcap_{i=1}^{n} V_{(p_i, r_i)}$, $p_i \in \mathscr{P}$ and $r_i > 0$ (the integer n is not fixed).

By an operator on X we mean a mapping of X into itself.

Theorem 3.1. Let T be an operator on X such that for any V in $\mathscr{V}$ and x, y in X,

$$(Tx, Ty) \in \alpha V\,, \qquad \text{if} \quad (x, y) \in V\,,$$

where $0 < \alpha < 1$. Then T has a unique fixed point in X.

Proof. Let x_0 be an arbitrary but fixed point of X. Define the sequence $\{x_n\}$ in X by

$$x_n = Tx_{n-1}\,, \qquad (n=1, 2, \cdots)\,.$$

Let V be any member of $\mathscr{V}$. Choose a positive number λ such that

$$(x_0, x_1) \in \lambda V = W\,, \qquad \text{say.}$$

Then

$$W \in \mathscr{V} \quad \text{and} \quad \mu W \in \mathscr{V} \quad \text{for any} \quad \mu > 0\,.$$

We have

$$(x_1, x_2) = (Tx_0, Tx_1) \in \alpha W\,,$$

$$(x_2, x_3) = (Tx_1, Tx_2) \in \alpha(\alpha W) = \alpha^2 W\,,$$

and by induction

$$(x_n, x_{n+1}) \in \alpha^n W\,.$$

Let n and m $(>n)$ be any two positive integers. Then since

$$(x_n, x_{n+1}) \in \alpha^n W \quad \text{and} \quad (x_{n+1}, x_{n+2}) \in \alpha^{n+1} W ,$$

we get

$$(x_n, x_{n+2}) \in \alpha^n W \circ \alpha^{n+1} W \subset (\alpha^n + \alpha^{n+1}) W . \qquad \text{[By Lemma 2.4]}$$

Similarly we have

$$(x_n, x_{n+3}) \in (\alpha^n + \alpha^{n+1} + \alpha^{n+2}) W ,$$

and proceeding in this way we obtain

$$(x_n, x_m) \in (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) W \subset \frac{\alpha^n}{1-\alpha} W = \frac{\lambda\alpha^n}{1-\alpha} V .$$

Since $0<\alpha<1$, we can choose a positive integer n_0 such that $\lambda\alpha^n/(1-\alpha)<1$ when $n \geqslant n_0$. Then $(x_n, x_m) \in V$ when $n \geqslant n_0$. Thus $\{x_n\}$ is a Cauchy sequence.

Since X is sequentially complete, there is a point ξ in X such that

$$\xi = \operatorname*{Lt}_{n\to\alpha} x_n .$$

From the given condition it is obvious that T is continuous. So

$$T(x_n) \to T(\xi) \qquad \text{as} \quad n \to \infty ,$$

that is,

$$x_{n+1} \to T(\xi) .$$

Since X is a Hausdorff space, $\xi = T(\xi)$. Let η be a point in X such that $\eta = T(\eta)$. Take any V in $\mathscr{V}$. Choose $\lambda>0$ such that

$$(\xi, \eta) \in \lambda V = W , \qquad \text{say.}$$

Then

$$(\xi, \eta) = (T(\xi), T(\eta)) \in \alpha W .$$

This gives that

$$(\xi, \eta) \in \alpha(\alpha W) = \alpha^2 W ,$$

and after n steps we obtain

$$(\xi, \eta) \in \alpha^n W = (\lambda\alpha^n) V .$$

Choose n so large that $\lambda\alpha^n<1$. Then

$$(\xi, \eta) \in V .$$

Since V is arbitarary, it follows that $\xi=\eta$. This completes the proof.

Theorem 3.2. Let T_1 and T_2 be two operators on X such that for any two members V_1, V_2 in $\mathscr{V}$ and x, y in X,

$$(T_1 x, T_2 y) \in \alpha V_1 \circ \beta V_2 ,$$

if $(x, T_1x) \in V_1$ and $(y, T_2y) \in V_2$ where α, β are independent of x, y, V_1, V_2 and $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. Then T_1 and T_2 have a unique common fixed point.

Proof. Let x_0 be an arbitrary but fixed point of X. Define the sequence $\{x_n\}$ in X as follows.

$$x_1=T_1x_0, \quad x_2=T_2x_1, \quad x_3=T_1x_2, \quad x_4=T_2x_3, \cdots ,$$

We prove the theorem by the following steps.

(I) The sequence $\{x_n\}$ converges to a point ξ in X. Let V be any member of $\mathscr{V}$. Denote by p the Minkowski's pseudometric of V. Let x, y be any two points of X. Write $p(x, T_1x)=r_1$ and $p(y, T_2y)=r_2$ and take $\varepsilon>0$. Then

$$(x, T_1x) \in (r_1+\varepsilon) V \quad \text{and} \quad (y, T_2y) \in (r_2+\varepsilon) V .$$

So by the given condition we have

$$(T_1x, T_2y) \in \alpha(r_1+\varepsilon) V \circ \beta(r_2+\varepsilon) V .$$

By Lemma 2.3 we have

$$\begin{aligned} p(T_1x, T_2y) &> \alpha(r_1+\varepsilon)+\beta(r_2+\varepsilon) \\ &= \alpha r_1+\beta r_2+(\alpha+\beta)\varepsilon . \end{aligned}$$

Since $\varepsilon>0$ is arbitrary,

$$p(T_1x, T_2y) \leqslant \alpha p(x, T_1x)+\beta p(y, T_2y) . \tag{4}$$

Now take any positive number λ with

$$\lambda \geqslant p(x_0, x_1) .$$

We have

$$\begin{aligned} p(x_1, x_2) &= p(T_1x_0, T_2x_1) \\ &\leqslant \alpha p(x_0, T_1x_0)+\beta p(x_1, T_2x_1) \qquad \text{[By (4)]} \\ &= \alpha p(x_0, x_1)+\beta p(x_1, x_2) . \end{aligned}$$

$$\begin{aligned} \therefore \quad p(x_1, x_2) &\leqslant \frac{\alpha}{1-\beta} p(x_0, x_1) \\ &\leqslant \frac{\alpha\lambda}{1-\beta} . \end{aligned}$$

$$\begin{aligned} p(x_2, x_3) &= p(T_2x_1, T_1x_2) \\ &= p(T_1x_2, T_2x_1) \\ &\leqslant \alpha p(x_2, T_1x_2)+\beta p(x_1, T_2x_1) \\ &= \alpha p(x_2, x_3)+\beta p(x_1, x_2) . \end{aligned}$$

$$\therefore \quad p(x_2, x_3) \leqslant \frac{\beta}{1-\alpha} p(x_1, x_2)$$
$$\leqslant \lambda \cdot \frac{\alpha}{1-\beta} \cdot \frac{\beta}{1-\alpha} .$$

By induction

$$p(x_{2n-1}, x_{2n}) \leqslant \lambda \left(\frac{\alpha}{1-\beta}\right)^n \left(\frac{\beta}{1-\alpha}\right)^{n-1}$$

and

$$p(x_{2n}, x_{2n+1}) \leqslant \lambda \left(\frac{\alpha}{1-\beta}\right)^n \left(\frac{\beta}{1-\alpha}\right)^n .$$

Write $\mu=\alpha/(1-\beta) \cdot \beta/(1-\alpha)$. Since $\alpha+\beta<1$, we have $0<\mu<1$. Take any two positive integers n and m $(>n)$. Then

$$p(x_{2n}, x_{2m}) \leqslant p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) + \cdots + p(x_{2m-1}, x_{2m})$$
$$\leqslant \lambda \left[\mu^n + \frac{\alpha}{1-\beta} \mu^n + \mu^{n+1} + \frac{\alpha}{1-\beta} \mu^{n+1} + \cdots + \frac{\alpha}{1-\beta} \mu^{m-1} \right]$$
$$= \lambda \left(1 + \frac{\alpha}{1-\beta}\right) [\mu^n + \mu^{n+1} + \cdots + \mu^{m-1}]$$
$$< \frac{1+\alpha-\beta}{1-\beta} \cdot \frac{\lambda \mu^n}{1-\mu} .$$

Choose a positive integer n_0 such that

$$\frac{1+\alpha-\beta}{1-\beta} \cdot \frac{\lambda \mu^n}{1-\mu} < 1 , \quad \text{when} \quad n \geqslant n_0 .$$

Then

$$p(x_{2n}, x_{2m}) < 1 \quad \text{for} \quad m > n \geqslant n_0 .$$

This gives that

$$(x_{2n}, x_{2m}) \in V , \quad \text{when} \quad m > n \geqslant n_0 .$$

Therefore $\{x_{2n}\}$ is a Cauchy sequence in X. Since X is sequentially complete, there is a point ξ in X such that

$$\xi = \operatorname*{Lt}_{n \to \alpha} x_{2n} .$$

Take V and p as above. Let n be any positive integer. Then

$$p(\xi, x_{2n+1}) \leqslant p(\xi, x_{2n}) + p(x_{2n}, x_{2n+1})$$
$$\leqslant p(\xi, x_{2n}) + \lambda \mu^n \to 0 \quad \text{as} \quad n \to \infty .$$

This gives that $p(\xi, x_{2n})<1$ if $n \geqslant n_1$ where n_1 is some positive integer. So

$$(\xi, x_{2n+1}) \in V \quad \text{when} \quad n \geqslant n_1 .$$

Thus $\{x_{2n+1}\}$ also converges to ξ. Hence $\{x_n\}$ converges to ξ.

(II) ξ is a common fixed point of T_1 and T_2. Let V be any member of $\mathscr{V}$. Denote by p the Minkowski's pseudometric of V. For any positive integer n we have

$$p(\xi, T_1\xi) \leqslant p(\xi, x_{2n}) + p(T_2x_{2n-1}, T_1(\xi))$$
$$\leqslant p(\xi, x_{2n}) + \alpha p(\xi, T_1\xi) + \beta p(x_{2n-1}, x_{2n}) . \qquad \text{[By (4)]}$$

Letting $n \to \infty$ we obtain

$$p(\xi, T_1\xi) \leqslant \alpha p(\xi, T_1\xi) .$$

Since $0<\alpha<1$, we get $p(\xi, T_1\xi)=0$. So

$$(\xi, T_1\xi) \in V .$$

V being arbitrary and X being a Hausdorff space we have

$$\xi = T_1\xi .$$

Similarly we can show that $\xi = T_2\xi$.

(III) ξ is the unique common fixed point of T_1 and T_2. Let η be a point in X with $\eta = T_1\eta$. Take any member V of $\mathscr{V}$. Then since

$$(\eta, T_1\eta) = (\eta, \eta) \in V$$

and

$$(\xi, T_2\xi) = (\xi, \xi) \in V ,$$

we have

$$(\eta, \xi) \in \alpha V \circ \beta V \subset (\alpha+\beta) V \subset V .$$

Since V is arbitrary it follows that $\eta = \xi$. Similarly if $\eta \in X$ and $\eta = T_2\eta$, then $\eta = \xi$. This completes the proof of the theorem.

Corollary 3.2.1. Let T_1 and T_2 be two operators and p, q be two positive integers such that for any V_1, V_2 in $\mathscr{V}$ and x, y in X

$$(T_1^p, T_2^q) \in \alpha V_1 \circ \beta V_2 ,$$

if $(x, T_1^p x) \in V_1$ and $(y, T_2^q y) \in V_2$ where α, β are independent of x, y, V_1, V_2 and $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. Then T_1, T_2 have a unique common fixed point.

Theorem 3.3. Let T_1 and T_2 be two operators on X such that for any V_1, V_2 in $\mathscr{V}$ and x, y in X

$$(T_1x, T_2y) \in \alpha V_1 \circ \beta V_2 ,$$

if $(y, T_1x) \in V_1$ and $(x, T_2y) \in V_2$ where α, β are independent of x, y, V_1, V_2 and $\alpha>0$, $\beta>0$ $\alpha+\beta<1$. Then T_1 and T_2 have a unique common fixed point.

The proof is similar to that of Th. 3.2.

Corollary 3.3.1. Let T_1 and T_2 be two operators on X and p, q be two positive integers such that for any V_1, V_2 in $\mathscr{V}$ and x, y in X

$$(T_1^p x, T_2^q y) \in \alpha V_1 \circ \beta V_2 ,$$

if $(y, T_1^p x) \in V_1$ and $(x, T_2^q y) \in V_2$, where α, β are independent of x, y, V_1, V_2, and $\alpha>0$, $\beta>0$, $\alpha+\beta<1$. Then T_1 and T_2 have a unique common fixed point.

Theorem 3.4. Let T_1 and T_2 be two operators on X such that for any three members V_1, V_2, V_3 in $\mathscr{V}$ and x, y in X

$$(T_1 x, y) \in \alpha V_1 \circ \beta V_2 \circ \gamma V_3 ,$$

if $(x, T_1 x) \in V_1$, $(x, y) \in V_2$, $(y, T_2 y) \in V_3$ where α, β, γ are independent of x, y, V_1, V_2, V_3 and $\alpha>0$, $\beta>0$, $\gamma>0$, $\alpha+\beta+\gamma<1$. Then T_1 and T_2 have a unique common fixed point.

Proof. Let x_0 be an arbitrary but fixed point of X. Define the sequence $\{x_n\}$ and take V and p as in the proof of Th. 3.2. Write

$$p(x, T_1 x)=r_1, \quad p(x, y)=r_2, \quad p(y, T_2 y)=r_3 ,$$

and take $\varepsilon>0$. Then

$$(x, T_1 x) \in (r_1+\varepsilon) V, \quad (x, y) \in (r_2+\varepsilon) V, \quad (y, T_2 y) \in (r_3+\varepsilon) V .$$

So by the given condition we have

$$(T_1 x, T_2 y) \in \alpha(r_1+\varepsilon) V \circ \beta(r_2+\varepsilon) V \circ \gamma(r_3+\varepsilon) V .$$

By Note 2.1,

$$\begin{aligned} p(T_1 x, T_2 y) &< \alpha(r_1+\varepsilon)+\beta(r_2+\varepsilon)+\gamma(r_3+\varepsilon) \\ &= \alpha r_1+\beta r_2+\gamma r_3+(\alpha+\beta+\gamma)\varepsilon . \end{aligned}$$

Since $\varepsilon>0$ is arbitrary,

$$p(T_1 x, T_2 y) \leqslant \alpha p(x, T_1 x)+\beta p(x, y)+\gamma p(y, T_2 y) . \tag{5}$$

Now take a positive number λ with $\lambda \geqslant p(x_0, x_1)$. Then

$$\begin{aligned} p(x_1, x_2) &= p(T_1 x_0, T_2 x_1) \\ &\leqslant \alpha p(x_0, T_1 x_0)+\beta p(x_0, x_1)+\gamma p(x, T_2 x_1) \\ &\leqslant \alpha p(x_0, x_1)+\beta p(x_0, x_1)+\gamma p(x_1, x_2) , \qquad \text{[By (5)]} \end{aligned}$$

or

$$\begin{aligned} p(x_1, x_2) &\leqslant \frac{\alpha+\beta}{1-\gamma} p(x_0, x_1) \\ &\leqslant \lambda \frac{\alpha+\beta}{1-\gamma} , \end{aligned}$$

$$\begin{aligned}p(x_2, x_3) &= p(T_1x_2, T_2x_1)\\ &\leqslant \alpha p(T_1x_2, x_2)+\beta p(x_1, x_2)+\gamma p(x_1, T_2x_1)\\ &= \alpha p(x_2, x_3)+\beta p(x_1, x_2)+\gamma p(x_1, x_2)\ ,\end{aligned}$$

or

$$\begin{aligned}p(x_2, x_3) &\leqslant \frac{\beta+\gamma}{1-\alpha}p(x_1, x_2)\\ &\leqslant \lambda\frac{(\alpha+\beta)(\beta+\gamma)}{(1-\gamma)(1-\alpha)}\ ,\end{aligned}$$

$$\begin{aligned}p(x_3, x_4) &= p(T_1x_2, T_2x_3)\\ &\leqslant \alpha p(T_1x_2, x_2)+\beta p(x_2, x_3)+\gamma p(x_3, T_2x_3)\\ &= \alpha p(x_2, x_3)+\beta p(x_2, x_3)+\gamma p(x_3, x_4)\ ,\end{aligned}$$

or

$$\begin{aligned}p(x_3, x_4) &\leqslant \frac{\alpha+\beta}{1-\gamma}p(x_2, x_3)\\ &\leqslant \lambda\left(\frac{\alpha+\beta}{1-\gamma}\right)^2\left(\frac{\beta+\gamma}{1-\alpha}\right).\end{aligned}$$

By induction,

$$p(x_{2n-1}, x_{2n})\leqslant\lambda\left(\frac{\alpha+\beta}{1-\gamma}\right)^n\left(\frac{\beta+\gamma}{1-\alpha}\right)^{n-1}$$

and

$$p(x_{2n}, x_{2n+1})\leqslant\lambda\left(\frac{\alpha+\beta}{1-\gamma}\right)^n\left(\frac{\beta+\gamma}{1-\alpha}\right)^n .$$

Write $\mu=(\alpha+\beta)/(1-\gamma)\cdot(\beta+\gamma)/(1-\alpha)$. Then $0<\mu<1$, as $\alpha+\beta+\gamma<1$. Then we complete the proof as in Th. 3.2.

Corollary 3.4.1. Let T_1 and T_2 be two operators on X and p, q be two positive integers such that for any three members V_1, V_2, V_3 in $\mathscr{V}$ and x, y in X,

$$(T_1^px, T_2^qy)\in\alpha V_1\circ\beta V_2\circ\gamma V_3\ ,$$

if $(x, T_1^px)\in V_1$, $(x, y)\in V_2$, $(y, T_2^qy)\in V_3$ where α, β, γ are independent of x, y, V_1, V_2, V_3 and $\alpha>0$, $\beta>0$, $\gamma>0$, $\alpha+\beta+\gamma<1$. Then T_1 and T_2 have a unique common fixed point.

Theorem 3.5. Let T_1 and T_2 be two operators on X such that for any three members V_1, V_2, V_3 in $\mathscr{V}$ and x, y in X

$$(T_1x, T_2y)\in\alpha V_1\circ\beta V_2\circ\gamma V_3\ ,$$

if $(y, T_1x)\in V_1$, $(x, y)\in V_2$, $(x, T_2y)\in V_3$ where α, β, γ are independent of x, y,

V_1, V_2, V_3 and $\alpha>0$, $\beta>0$, $\gamma>0$, $\alpha+\beta+\gamma<1$. Then T_1 and T_2 have a unique common fixed point.

The proof is similar to that of Th. 3.4.

Corollary 3.5.1 Let T_1 and T_2 be two operators on X and p, q be two positive integers such that for any three members V_1, V_2, V_3 in $\mathscr{V}$ and x, y in X,

$$(T_1^p x, T_2^q y) \in \alpha V_1 \circ \beta V_2 \circ \gamma V_3 ,$$

if $(y, T_1^p x) \in V_1$, $(x, y) \in V_2$, $(x, T_2^q y) \in V_3$ where α, β, γ are independent of x, y, V_1, V_2, V_3 and $\alpha>0$, $\beta>0$, $\gamma>0$, $\alpha+\beta+\gamma<1$.

Then T_1 and T_2 have a unique common fixed point.

I am much indebted to Dr. P. C. Bhakta Jadavpur University, for his kind help and guidance in the preparation of this paper.

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