TOPOLOGICAL-GROUP-VALUED MEASURES

By

SACHIO OHBA

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1. Introduction

In [4] *Takahashi* has proved the following theorem. Let S be a set, R a ring of subsets of $S, \varphi(R)$ the δ-ring generated by R, G a complete, Hausdorff, commutative topological group and $m: R \rightarrow G$ a measure. If m has the $B-V$ property, then m can be extended to uniquely a measure $m_1: \varphi(R) \rightarrow G$.

He also raised the following problem: Whether this theorem remains valid if the $B-V$ property of m is replaced by the locally s-boundedness of m? In this paper we shall give the positive answer for the problem. These results are the extensions of [6].

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2. Extension theorems

Let S be a set, R a ring of subsets of S, $\varphi(R)$ the δ-ring (that is, a ring closed under countable intersection) generated by R, G a complete, Hausdorff, commutative topological group and $\boldsymbol{u}$ a base for neighborhoods of 0 in G, consisting of closed symmetric sets.

Difinition 1. A set function $m: R \rightarrow G$ is called a measure if for every sequence $\{E_n\}$ of mutually disjoint sets of R such that $E = \bigcup_{n=1}^{\infty} E_n \in R$ we have $m(E) = \sum_{n=1}^{\infty} m(E_n)$.

Definition 2. A measure $m: R \rightarrow G$ is called locally s-bounded if for every set $E \in R$ and every sequence $\{E_n\}$ of mutually disjoint sets of R with $\bigcup_{n=1}^{\infty} E_n \subset E$ we have $\lim_{n \to \infty} m(E_n) = 0$. It is easy to show that if R is a δ-ring, then every measure $m: R \rightarrow G$ is locally s-bounded.

Theorem 1. If $m: R \rightarrow G$ is locally s-bounded, then m can be extended to uniquely a measure $m_1: \varphi(R) \rightarrow G$.

Proof. Only the outline of the proof will be given here, since further complementation is quite easy (for example, see [3] or [6]).

We put $\tilde{R}=\{\bigcup_{n=1}^{\infty} E_n: E_n \in R$ $(n=1, 2, \cdots)$ and $\bigcup_{n=1}^{\infty} E_n \subset E$ for some set $E \in R\}$. Then we have following.

1°. For every set $E=\bigcup_{n=1}^{\infty} E_n \in \tilde{R}$ there exists a unique set function $\hat{m}: \tilde{R} \to G$ such that $\lim_{n\to\infty} m(\bigcup_{i=1}^{n} E_i)=\hat{m}(E)$.

We can prove in the same way as the proof of Lemma 1 of [4].

It follows that 2°. $E \in R \Longrightarrow \hat{m}(E)=m(E)$

$$E, F \in \tilde{R} \Longrightarrow \hat{m}(E \cup F)+\hat{m}(E \cap F)=\hat{m}(E)+\hat{m}(F)\ .$$

3°. For every set $E \in \tilde{R}$ and every neighborhood $U \in \boldsymbol{u}$ there exists a set $A \in R$ such that $A \subset E$ and for every set $B \in R$ with $B \subset E-A$ we have $m(B) \in U$. In case $E \in R$, it is obvious. Suppose that $E \in \tilde{R}$ and $E \notin R$. If it were false, then there exist a set $E \in \tilde{R}$ and a neighborhood $U \in \boldsymbol{u}$ such that for every set $A \in R$ with $A \subset E$ there exists a set $B \in R$ with $B \subset E-A$ and $m(B) \notin U$. We put $A=\phi$. Then there exists a set $B_1 \in R$ such that $B_1 \subset E$ and $m(B_1) \notin U$. Next, we put $A=B_1$. Then there exists a set $B_2 \in R$ such that $B_2 \subset E-B_1$ and $m(B_2) \notin U$. Hence $B_1 \cap B_2=\phi$. By induction there exists a sequence $\{B_n\}$ of mutually disjoint sets of R such that $B_n \subset B_1 \cup B_2 \cup \cdots \cup B_{n-1}$ and $m(B_n) \notin U$. By the locally s-boundedness of m we have $\lim_{n\to\infty} m(B_n)=0$. Therefore we have a contradiction.

We put $H=\{A \subset S: A \subset E$ for some set $E \in R\}$.

4°. For any fixed set $A \in H$, $\Gamma(A)=\{B \in \tilde{R}: A \subset B\}$ is a directed set, when we write $B_1 \leqq B_2$ if and only if $B_1 \supset B_2$, for $B_1, B_2 \in \Gamma(A)$. The generalized sequence $\{\hat{m}(B): B \in \Gamma(A)\}$ is a Cauchy net in G.

We can prove in the same way as the proof of Lemma 5.1 of [3].

5°. There exists a unique set function $m^*: H \to G$ such that for every set $A \in H$, every set $B \in \Gamma(A)$ and every neighborhood $U \in \boldsymbol{u}$ there exists a set $C \in \Gamma(A)$ with $C \subset B$ and $\hat{m}(C)-m^*(A) \in U$.

By 4° and the completeness of G we put $m^*(A)=\lim\{\hat{m}(B): B \in \Gamma(A)\}$. Then m^* has the above property. It follows that $A \in R \Longrightarrow m^*(A)=\hat{m}(A)=m(A)$ and $A \in \tilde{R} \Longrightarrow m^*(A)=\hat{m}(A)$.

Definition 3. A set $A \subset S$ is measurable if and only if for every set $M \in H$, $m^*(M)=m^*(M \cap A)+m^*(M-A)$.

Let Σ be the class of measurable sets and $\Sigma_0=\Sigma \cap H$. Then we have the following properties.

6°. (1) Σ is an algebra of subsets of S.

(2) $R \subset \Sigma_0$ and the set function m^* is finitely additive on Σ_0.

We can prove in the same way as the proof of Lemma 5.4 of [3].

7°. $A_n,\ A\in R,\ A_n\uparrow A \Longrightarrow \lim_{n\to\infty} \hat{m}(A_n)=\hat{m}(A)$.

We can prove in the same way as the proof of Lemma 4.4 of [3].

8°. $\tilde{R}\subset\Sigma_0$.

We can prove in the same way as the proof of Lemma 5.5 of [3].

9°. $A_n,\ A\in H,\ A_n\uparrow A \Longrightarrow \lim_{n\to\infty} m^*(A_n)=m^*(A)$.

We can prove in the same way as the proof of Theorem 5.6 of [3].

10°. For every set $M\in H$ and every sequence $\{A_n\}$ of mutually disjoint sets of Σ we have $m^*(M\cap \bigcup_{n=1}^{\infty} A_n)=\sum_{n=1}^{\infty} m^*(M\cap A_n)$.

For disjoint sets $A, B\in\Sigma$ we have $m^*(M\cap(A\cup B))=m^*(M\cap(A\cup B)\cap A)+m^*(M\cap(A\cup B)-A)=m^*(M\cap A)+m^*(M\cap B)$. Therefore for finite sequence $\{A_n\}$ it is obvious.

Let $\{A_n\}$ be an infinite sequence. We put $B_n=M\cap\bigcup_{i=1}^{n} A_i$. Then $B_n\in H$ and $B_n\uparrow M\cap\bigcup_{n=1}^{\infty} A_n$. Therefore by 9° $m^*(M\cap\bigcup_{n=1}^{\infty} A_n)=\lim_{n\to\infty} m^*(B_n)=\lim_{n\to\infty}\sum_{i=1}^{n} m^*(M\cap A_i)$ $=\sum_{n=1}^{\infty} m^*(M\cap A_n)$.

11°. The set function m^* is countably additive on Σ_0.

By 10° it is obvious.

12°. Σ is a σ-algebra of subsets of S.

Let $\{A_n\}$ be a sequence of Σ such that $A_n\uparrow A$. For every set $M\in H$ we have

$$m^*(M)=\lim_{n\to\infty} m^*((M\cap A_n)\cup(M-A))=\mathrm{mil}_{n\to\infty}\{m^*(M\cap A_n)+m^*(M-A)\}$$
$$=m^*(M\cap A)+m^*(M-A)\ .$$

Then we have $A\in\Sigma$.

13°. Σ_0 is a δ-ring of subsets of S.

It is obvious.

Since $R\subset\Sigma_0$ and Σ_0 is δ-ring we have $\varphi(R)\subset\Sigma_0$. Let m_1 be the restriction of m^* to $\varphi(R)$. Then $m_1:\varphi(R)\to G$ is a measure such that for every set $A\in R$ $m_1(A)=m(A)$. The uniqueness of m_1 is obvious by §2. Proposition 6 of [1].

Definition 4. A measure $m:R\to G$ has the $B-V$ property if and only if for every set $E\in R$ and every neighborhood $U\in\boldsymbol{u}$ there exists a positive integer N such that, for every finite sequence $\{E_i\}_{1\leq i\leq N}$ of mutually disjoint sets of R with $\bigcup_{i=1}^{N} E_i\subset E$ there exists a positive integer $i_0(1\leqq i_0\leqq N)$ such that $m(E_{i_0})\in U$.

Theorem 2. If $m:R\to G$ have $B-V$ property, then m is locally s-bounded.

Proof. It is obvious.

For every set $V\subset G$ and every positive integer k we put

$$kV=\{\sum_{i=1}^{k} y_i : y_i \in V, \quad i=1,2,\cdots,k\}.$$

Definition 5. A set $K\subset G$ is called bounded if for every neighborhood $U\in \boldsymbol{u}$ there exists a positive integer k such that $K\subset kU$.

Theorem 3. Suppose that G has the following properties:

(1) every singleton set in G is bounded.

(2) if $\{a_n\}$ is a sequence in G such that $a_n \notin U$ $(n=1,2,\cdots)$ for some neighborhood $U\in \boldsymbol{u}$, then the set $\{\sum_r a_{n_r} : \{a_{n_r}\}$ is a finite subsequence of $\{a_n\}\}$ is not bounded.

Then the following statements are equivalent.

(A) m is locally s-bounded.

(B) for every set $E\in R$ the set $\{m(F): F\subset E,\ F\in R\}$ is bounded.

Proof. The proof of $(A)\Longrightarrow(B)$ is obvious by Theorem 3.2.1 of [5].

$(B)\Longrightarrow(A)$. If it were false, then there exist a set $A\in R$, a sequence $\{A_n\}$ of mutually disjoint sets of R with $A_n\subset A$ and a neighborhood $U\in \boldsymbol{u}$ such that $m(A_n)\notin U$ for all n. By (2) the set $\{\sum_r m(A_{n_r}) : \{m(A_{n_r})\}$ is a finite subsequence of $\{m(A_n)\}\}$ is not bounded. Therefore we have a contradiction.

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Kanagawa University
Rokkaku-bashi, Kanagawa-ku
Yokohama, Japan.