SINGULAR BLOCK BUNDLES

By

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(Received April 25, 1973)

1. Introduction

It is well-known that, by the famous combinatorial prebundle theory [3] and the block bundle theory [4], we are able to study many manifolds with some lower dimensional manifolds as their spines from their spines. Some manifolds, however, do not have any manifolds as spines. The main purpose of this paper is to find something like above bundle structures for manifolds with some polyhedra as their spines. Such bundle-like structures are called *singular block bundles* (block bundles with singularity). Briefly speaking, the base spaces of singular block bundles are fake manifolds which are not necessarily manifolds) and the total spaces of them are manifolds. Hence, of course, the fibers of the singular block bundles are a bit complicated. One of the reasons why we consider about such singular block bundles is to pick up the "standard collapsings" among so many collapsings of manifolds to the given spines to realize the "inverse images" of the collapsings with respect to some sub-polyhedra of the spines geometrically. This problem is raised by *H. Noguchi* in our seminar held by All Japan Combinatorial Topology Study Group.

In §2, some well-known propositions are stated.

In §3, we define the blocks which are the same objects as those of combinatorial prebundles and obtain some natural properties of blocks.

The n-dimensional fiber-set Φ^n is introduced in §4. Φ^n is the set consisting of three polphedra J^n, Y^n and X^n, each of which is a homogeneous n-dimensional polyhedron with simple shape. When we define the singular block bundles later, the fibers of the blocks are chosen in Φ^n.

The most difficult problem we have to deal with lies in §5. We define fake manifolds which extends the concept of fake surfaces introduced in [2] naturally. The fake surfaces are fake 2-manifold in this definition. The problem mentioned above is to characterize a pair of simplexes of a simplicial complex whose underlying polyhedron is a fake manifold. Remark that we assume $\mathfrak{S}_7(P)=\emptyset$ for any fake manifold P throughout this paper (for the numbering of the singularity of a fake manifold, see Definition 6, it is a bit different from one in [2]). This assumption may be allowed, because it does not give any restrictions on manifolds which we want to deal with as total spaces.

In § 6, we can define the singular block bundles over fake manifolds. The blocks are determined according to which singularity of fake manifolds the simplexes (the base of blocks) are contained in. Theorem 1 states a relation between the combinatorial prebundles, the block bundles and the singular block bundles. And, we obtain the required property of the singular block bundles in Theorem 2.

Theorem 2. *The total space of a singular block bundle is a manifold which collapses to the base space.*

In § 7, we study about the regular neighborhoods of "locally unknotted fake manifolds" in manifolds. And, we obtain the following theorem.

Theorem 4. *Let P be a locally unknotted fake p-manifold properly embedded in a q-manifold V. Then, the regular neighborhood of P in V meeting the boundary regularly is a singular block bundle over P with fiber-set Φ^{q-p}.*

Furthermore, for 3-manifold, we obtain the following.

Theorem 5. *Let V be a 3-manifold with boundary. Then, there exists a closed fake surface P such that V is a singular block bundle over P with fiber-set Φ^1.*

The author thanks all the members of All Japan Combinatorial Topology Study Group for their kind suggestions and discussions.

2. Preliminaries

In this section, some elementary materials are stated. And, for the other general properties, refer [5].

For a polyhedron P, we define the *boundary* $\dot{P}$ of P to be the union of the (closed) free faces of P with respect to the polyhedral collapsings of P, that is $\dot{P}$ is the union of the balls contained in P from which we can collapse P. And, the *interior* $\mathring{P}$ of P is defined by $\mathring{P}=P-\dot{P}$. We say that a polyhedron P is *closed* when the boundary $\dot{P}$ is empty.

For a sub-polyhedron Q of a polyhedron P, we say that Q is *proper* in P when Q satisfies $Q\cap\dot{P}=\dot{Q}$.

For a simplicial complex K, we define the *boundary* $\dot{K}$ to be the union of the (closed) free faces with respect to the simplicial collapsings of K and their faces. Then, by the same way, we can define the interior $\mathring{K}$ of K, and the others, that is, the *closed* simplicial complex and the *proper* sub-complex.

Here, we write two well-known propositions.

Proposition 1. *Let P and K be a polyhedron and a simplicial complex, respectively. Then, the boundaries $\dot{P}$ and $\dot{K}$ are a sub-polyhedron of P and a sub-complex of K, respectively.*

For a simplicial complex K, let $|K|$ denote the underlying polyhedron of K. Then, we have the following.

Proposition 2. *Let K be a simplicial complex. Then, we obtain $|\dot{K}|=(|K|)^{\cdot}$, that is, the underlying polyhedron of the boundary is the bounnary of the underlying polyhedron.*

3. The blocks.

As is usual, we start with the definition of the blocks, which are the same objects as those introduced in the combinatorial pre-bundles (cf. [3]). And furthermore, the sub-blocks and the restricted blocks of the blocks are also defined.

Definition 1. (The blocks) Let F be a polyhedron and A an n-simplex. We define the ***block*** F_A over A with fiber F to be the polyhedron $A\times F$.

In the above definition, it should be understood that the block F_A is empty when either the simplex A or the fiber F is empty.

Definition 2. (The sub-blocks) Let a block F_A be given and let G be a sub-polyhedron of F. We define the ***sub-block*** $(F|G)_A$ of the block F_A with respect to G to be the sub-polyhedron of F_A determined by

$$(F_A, (F|G)_A)=(A\times F, A\times G) .$$

The sub-block $(F|G)_A$ is said to be ***proper*** in the ***main*** block F_A when G is a proper sub-polyhedron of F.

Definition 3. (The restricted blocks) Let a block F_A be given and let B be a face of A. We define the ***restricted block*** $(F_A|B)$ of the block F_A on B to be the sub-polyhedron of F_A determined by

$$(F_A, (F_A|B))=(A\times F, B\times F) .$$

Note that the sub-blocks and the restricted ones are embedded in the respective main blocks by the natural inclusion maps. And, from the definitions, it is clear that the sub-blocks and the restricted ones are blocks by themselves. Accordingly, the sub-block $(F|G)_A$ and the restricted one $(F_A|B)$ are sometimes denoted by G_A

and F_B respectively when there may be no confusion.

In the following, the boundary block of a block is introduced.

Definition 4. Suppose that a block F_A is given. The special sub-block $(F|\dot{F})_A$ is called the *boundary block* of F_A and is always written $\dot{F}_A$.

As is clearly seen, there is a difference between the boundary block and the boundary of the block. And hence the boundary of the block F_A is denoted by $(F_A)^{\cdot}$.

Here, we state some easy lemmas about the concepts defined in this section.

Lemma 1. *Let $(F|G_1)_A$ and $(F|G_2)_A$ be two sub-blocks of a block F_A. Put $G_1 \cap G_2 = G_3$. Then, we obtain*

$$(G_1)_A \cap (G_2)_A = (G_3)_A ,$$

that is, the intersection of two sub-blocks is again a sub-block.

Lemma 2. *Let $(F_A|B_1)$ and $(F_A|B_2)$ be two restricted blocks of a block F_A. Put $B_1 \cap B_2 = B_3$. Then, we obtain*

$$F_{B_1} \cap F_{B_2} = F_{B_3} ,$$

that is, the intersection of two restricted blocks is again a restricted block.

Now, the blocks have the natural maximal structures consisting of their sub-block and the restricted ones. And, the structures of the sub-blocks and the restricted ones are sub-structures of those of the main blocks.

Let us continue the lemmas. The following one shows the relation between the sub-block of the restricted block and the restricted block of the sub-block of a block.

Lemma 3. *Let $(F|G)_A = G_A$ be a sub-block of a block F_A and let $(F|G)_B$ be a sub-block of the restricted block $(F_A|B) = F_B$ of a block F_A on B. Then, we obtain*

$$(F|G)_B = (G_A|B) .$$

In the following, we show the relation between the boundary block and the boundary of the block.

Lemma 4. *Let F_A be a block. Then, we obtain*

$$(F_A)^{\cdot} = \dot{F}_A \cup \bigcup_B (F_A|B) ,$$

where B is a face of A and $B \neq A$.

4. The n-dimensional fiber-set Φ^n.

In this section, we introduce the concept of the n-dimensional fiber-set Φ^n consisting of three polyhedra each of which has a very simple shape as is seen in the definition and plays a very important role within our singular block bundles (defined later) because we choose the fibers of the blocks to be the sub-polyhedra of the elements of Φ^n.

Definition 5. (The n-dimensional fiber-set Φ^n) The set $\Phi^n=\{J^n, Y^n, X^n\}$ consisting of the three n-dimensional homogeneous polyhedra J^n, Y^n and X^n which are defined below is called the *n-dimensional fiber-set.*

(0) When $n=0$, the elements J^0, Y^0 and X^0 of Φ^0 are the sub-polyhedra of R^2 defined by

$$J^0=\{(-1, 0),\ (1, 0)\}\ ,$$

$$Y^0=\{(-1, 0),\ (0, -1),\ (1, 0)\}\ ,$$

$$X^0=\{(-1, 0),\ (0, -1),\ (0, 1),\ (1, 0)\}\ .$$

(1) When $n=1$, the elements J^1, Y^1 and X^1 of Φ^1 are the sub-polyhedra of R^2 defined by

$$J^1=o * J^0\ ,$$

$$Y^1=o * Y^0\ ,$$

$$X^1=o * X^0\ ,$$

where o denotes the origin of R^2 and the symbol $*$ means the "join". The common point o of the elements of Φ^1 is called the *center* of them (or Φ^1) and is written $o(F)$, where F is an element of Φ^1, or just o.

(2) When $n\geqq 2$, the element F^n of Φ^n is defined inductively by

$$F^n=F^1\times J^{n-1}\ ,$$

where, of course, F is either J or Y or X. The common point $(o(F^1), o(J^{n-1}))$ of the element F^n of Φ^n is called the *center* of F^n (or Φ^n) and is written $o(F^n)$ or $o(\Phi^n)$ or just o. And the sub-polyhedron $o(F^1)\times J^{n-1}$ of F^n is called the *core* of F^n or Φ^n and is written core (F^n) or core (Φ^n).

It is clear, from the definition, that Φ^0 contains neither the center nor the core, and, for Φ^1, it should be understood that the center and the core are the same.

In the rest of this paper, whenever we say a polyhedron F a fiber, F is always a sub-polyhedron of an element of Φ^n for some n. And, for a given fiber F, any sub-polyhedron G of F is called a *sub-fiber* of F.

In the following, we make same definitions about the special sub-fibers.

Definition 6. Let F be an element of Φ^n and let G be a sub-fiber of F.

(1) We say that G is ***strongly proper*** in F when G is proper in F and is an n-dimensional homogeneous polyhedron.

(2) We say that G is ***semi-proper*** in F when G is an n-ball obtained by taking the closure of a connected component of $F-\text{core}\,(F)$.

(3) We say that G is ***trivial*** in F when G is the core of F.

A sub-fiber G of an element F of Φ^n is said to be ***normal*** in F when G is either strongly proper or semi-proper or trivial or empty in F. And, we say that a sub-block $(F|G)_A$ of a block F_A is ***strongly proper*** or ***semi-proper*** or ***trivial*** in F_A according whether G is strongly proper or semi-proper or trivial in F. Similarly, when G is normal in F, we say that the sub-block $(F|G)_A$ is ***normal*** in F_A.

Now, the following lemmas are trivial.

Lemma 5. *Let G_1 and G_2 be two normal sub-fibers of an element F of Φ^n. Then, the intersection $G_1 \cap G_2$ is again a normal sub-fiber of F.*

Hence, we obtain the following.

Lemma 6. *Let $(F|G_1)_A$ and $(F|G_2)_A$ be two normal sub-blocks of a block F_A with fiber F in Φ^n. Then, the intersection $(F|G_1)_A \cap (F|G_2)_A$ is again a normal sub-block of F_A.*

5. The base complexes.

In the first part of this section, we define the ***p-dimensional fake manifolds*** which is a generalization of the concept of the fake surfaces introduced in [2], that is, the fake surfaces are the fake 2-manifolds. And, we define the simplical fake manifolds, written SFM, naturally. Most of this section is devoted to characterize the relations between two simplexes of an SFM as a preparation to the definition of the singular block bundles in the next section in which the base complexes are limited only to the SFM's.

Let St_1 denote the standard p-ball in R^q $(q\geqq p+1)$ defined by

$$St_1=\{(x_1, \cdots, x_p, 0, \cdots, 0)\,|\,|x_i|\leqq 1\}\,.$$

And, let B_1, B_2 and B_3 be $(p-1)$-balls in St_1 defined by

$$B_1=\{(x_1, \cdots, x_{p-1}, 0, \cdots, 0) \mid |x_i| \leqq 1\} ,$$

$$B_2=\{(x_1, \cdots, x_{p-2}, 0, x_p, 0, \cdots, 0) \mid |x_i| \leqq 1\} ,$$

$$B_3=\{(x_1, \cdots, x_{p-1}, 0, \cdots, 0) \mid |x_i| \leqq 1, x_{p-1} \geqq 0\} .$$

Let us define three p-balls C_1, C_2 and C_3 in R^q by the followings.

$$C_1=\{(x, 0, x_{p+1}, 0, \cdots, 0) \mid x \in B_1, 0 \leqq x_{p+1} \leqq 1\} ,$$

$$C_2=\{(x, 0, x_{p+1}, 0, \cdots, 0) \mid x \in B_2, -1 \leqq x_{p+1} \leqq 0\} ,$$

$$C_3=\{(x, 0, x_{p+1}, 0, \cdots, 0) \mid x \in B_3, 0 \leqq x_{p+1} \leqq 1\} .$$

Define the three polyhedra St_i, $i=2, 3, 4$, as follows.

$$St_i=St_1 \cup \bigcup_{j=1}^{i-1} C_j , \qquad i=2, 3 .$$

$$St_4=St_1 \cup C_3 .$$

Let $\mathscr{S}^p$ denote the set of St_i, $i=1, \cdots, 4$.

These are, clearly, the p-dimensional cases of those descibed in Fig. 1 in [2].

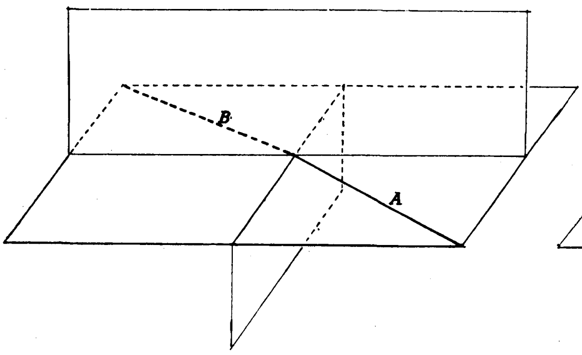

Fig. 1-1.

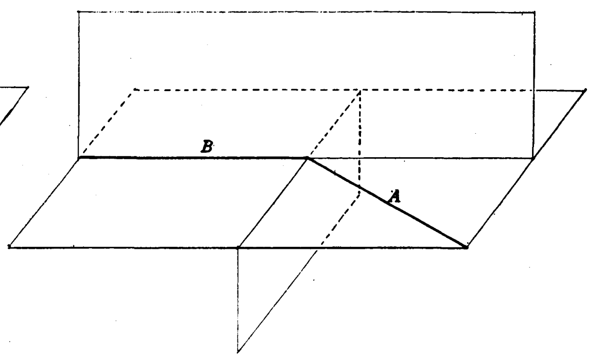

Fig. 1-2.

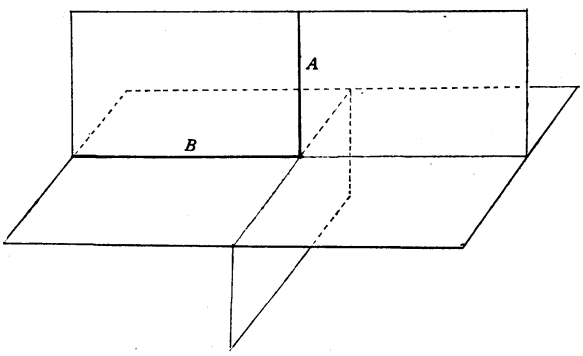

Fig. 1-3.

Definition 5. A polyhedron P is said to be a *fake p-manifold* if, for any point x of P, $st(x, P)$ belongs to the set $\mathscr{S}^p$.

Following [2], we define the i-th singularity for a fake p-manifold. Let x be a point of a fake manifold P. Then, for $i=1, 2, 3$, x is said to be of *type* i if $st(x, P)$ is St_i and x is contained in the interior of $st(x, P)$. And, x is said to be of *type* $i+3$, if $st(x, P)=St_i$ and x belongs to the boundary of $st(x, P)$, for $i=1, 2, 3$. When $st(x, P)$ is St_4, x is said to be of *type* 7.

Definition 6. For a fake manifold P, the closure of the set $\{x \in P | x$ is of type $i\}$ is called the *i-th singularity* of P and is written $\mathfrak{S}_i(P)$, $i=1, \cdots, 7$.

Remark. The numbering of the singularities of the fake surfaces is different from one of [2], that is, for a fake surface P, the 6-*th* singularity $\mathfrak{S}_6(P)$ of [2] is the 7-*th* singularity $\mathfrak{S}_7(P)$ in the above definition.

Now, we state two propositions concerning with the fake manifolds, which are easily proved from the definition.

Proposition 3. *Let P be a fake p-maifold. Then, we obtain the following.*

(1) *The i-th singularity $\mathfrak{S}_i(P)$ is a homogeneous sub-polyhedron of P, for $i=1, \cdots, 7$.*

(2) $\mathfrak{S}_1(P)=P$.

(3) *For $i=2, 3, 5, 6, 7$, $\mathfrak{S}_i(P)$ is contained in $\mathfrak{S}_2(P)$.*

(4) *When $\mathfrak{S}_2(P)$ is non-empty, $\mathfrak{S}_2(P)$ is of dimension $p-1$.*

(5) *When $\mathfrak{S}_3(P)$ is non-empty, then $\mathfrak{S}_3(P)$ is a $(p-2)$-manifold.*

Proof. The proofs are, as we mentioned, established by the standard way. And, here, we give just a proof of the condition (5) as an example. For any point x of $\mathfrak{S}_3(P)$, we have $st(x, P)=St_1 \cup C_1 \cup C_2$ from the definition. Then, it is easy to see $st(x, \mathfrak{S}_3(P))=C_1 \cap C_2$. Hence, $\mathfrak{S}_3(P)$ is a $(p-2)$-manifold, because $C_1 \cap C_2=B_1 \cap B_2$ is a $(p-2)$-ball.

Using the fact $st(x, \dot{P})=st(x, \dot{st}(x, P))$, we can prove the following proposition easily.

Proposition 4. *Let P be a fake p-manifold. Suppose that the boundary $\dot{P}$ is non-empty. Then, $\dot{P}$ is a fake $(p-1)$-manifold.*

Following [2], let $U(P)$ and $M(P)$ denote the 3-rd derived neighborhood of $\mathfrak{S}_2(P)$ in a fake manifold P and the closure of the complement of $U(P)$ in P, respectively. It is not difficult to see that the polyhedra $U(P)$ and $M(P)$ are independent from the choice of the triangulation of P. Of course, $M(P)$ is a p-manifold, if P is of dimension p. And note that $M(P)$ also denote the set of the connected components of the manifold $M(P)$.

We do not develope, here, the general theory of the fake manifolds any more. And, we go to the main problem in this section.

A simplical complex K is said to be a ***simplicial fake p-manifold***, denoted by p-SFM, if the underlying polyhedron of K, written $|K|$, is a (polyhedral) fake p-manifold. Then, for an SFM K, we obtain naturally the sub-complex $\mathfrak{S}_i(K)$ of K, called the ***i-th singularity*** of K, triangulating the *i-th* singularity $\mathfrak{S}_i(P)$ of the fake manifold $P=|K|$.

Let K be an SFM. And, let A be a simplex of K and C a face of A. By $D_i(A, C)$, we denote the closure of the union of the connected components of

$$|st(C, \mathfrak{S}_i(K))|-|st(C, \mathfrak{S}_{i+1}(K))| ,$$

each of whose closures contains A.

Definition 7. Let K be an SFM. Let A and B be two simplexes of K satisfying the condition $A\cap B=C\neq\emptyset$. We say that A and B belong to the ***same side*** (in K), when $D_1(A, C)=D_1(B, C)$.

For two simplexes A and B of an SFM K satisfying $A\cap B\neq\emptyset$, we say that A and B belong to the ***distinct sides*** (in K), when A and B do not belong to the same side (in K).

For the set of pairs consisting of two simplexes of an SFM belonging to the distinct sides, it is necessary to define the smaller sub-sets as follows.

Definition 8. Suppose that A and B are simplexes of an SFM K belonging to the distinct sides. Put $A\cap B=C$ and $\dim K=p$.

(1) We say that A and B are 1-*related* (in K), if $\dim(D_1(A, C)\cap D_1(B, C))=p-2$. (See Fig. 1-1)

(2) We say that A and B are 2-*related* (in K), if one of the following two conditions is satisfied. (See Fig. 1-2)

(2-a) $\dim(D_1(A, C)\cap D_1(B, C))=p-1$.

(2-b) $\dim(D_2(A, C)\cap D_2(B, C))=p-2$.

(3) We say that A and B are 3-*related* (in K), if A and B are neither 1- nor 2-related. (See Fig. 1-3).

Suppose that an SFM K is given. Let $\Omega(K)$ denote the set of pairs (A, B) consisting of two simplexes A and B of K with $A\cap B\neq\emptyset$. Here, we define four sub-sets $\Omega_i(K)$ of $\Omega(K)$, $i=1, \cdots, 4$, as follows—

(1) For $i=1, 2, 3$, $\Omega_i(K)$ consists of the element (A, B) such that A and B are i-related.

(2) $\Omega_4(K)$ consists of the elements (A, B) such that A and B belong to the same side.

Lemma 7. *Let K be an SFM and (A, B) an element of $\Omega_4(K)$. Then, both A and B are contained in either $K-\mathfrak{S}_2(K)$ or $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$ or $\mathfrak{S}_3(K)$.*

Proof. It is clear that there exist unique numbers i and j such that $\mathfrak{S}_i(K)-Q_{i+1}(K)$ contains A and $\mathfrak{S}_j(K)-Q_{j+1}(K)$ contains B, $i, j=1, 2, 3$, where $Q_{i+1}(K)=\mathfrak{S}_{i+1}(K)$ if $i=1, 2$, and $Q_4(K)=\emptyset$. And, we have to show $i=j$.

Case 1. Suppose $i=1$ and $j\geqq 2$. Then, $A\cap B=C$ is contained in $\mathfrak{S}_2(K)$. Let us consider

$$P=|st(C, K)|-|st(C, \mathfrak{S}_2(K))|\ .$$

Now, P has at least two connected components, just one of which, say E, contains $\mathring{A}$, because $K-\mathfrak{S}_2(K)$ contains A. Hence, we obtain $D_1(A, C)=\bar{E}$, that is, $D_1(A, C)$ is the closure of E. On the other hand, $D_1(B, C)$ contains at least two components of P, because B is contained in $\mathfrak{S}_2(K)$. Thus, we obtain $D_1(A, C)\neq D_1(B, C)$.

Case 2. Suppose $(i, j)=(2, 3)$. In this case, C is contained in $\mathfrak{S}_3(K)$ and hence P has six connected components, just three of which are contained in $D_1(A, C)$. On the other hand, we obtain $D_1(B, C)=P$. Hence, we see $D_1(A, C)\neq D_1(B, C)$. Thus, (A, B) can not be an element of $\Omega_4(K)$. This complete the proof.

By the similar argument to one used in the proof of Lemma 7, we have the following lemmas.

Lemma 8. *Let K be an SFM and (A, B) an element of $\Omega(K)$. Suppose that A and B are contained in $\mathfrak{S}_3(K)$. Then, (A, B) is an element of $\Omega_4(K)$.*

Proof. It is clear, because, putting $C=A\cap B$, we obtain $D_1(A, C)=st(C, K)=D_1(B, C)$.

Lemma 9. *Let K be an SFM. Suppose that A and B are simplexes of K and $\mathfrak{S}_i(K)-Q_{i+1}(K)$ contains A and B is contained in $\mathfrak{S}_j(K)-Q_{i+1}(K)$, $i, j=1, 2, 3$, where $Q_{i+1}(K)=\mathfrak{S}_{i+1}(K)$ for $i=1, 2$, and $Q_4(K)=\emptyset$. Put $C=A\cap B$.*

(1) *If (A, B) is an element of $\Omega_1(K)$, then $i=1=j$ and C is contained in $\mathfrak{S}_3(K)$.*

(2) *If (A, B) is an element of $\Omega_2(K)$, then $i\neq 3\neq j$ and C is contained in $\mathfrak{S}_k(K)$ where $k=\min(i, j)+1$.*

(3) *If (A, B) is an element of $\Omega_3(K)$, then $i\neq j$ and C is contained in $\mathfrak{S}_2(K)$.*

Here, we have a proposition.

Proposition 5. *Let K be an SFM. Then, we obtain the following.*

(1) $\Omega_i(K)\cap\Omega_j(K)=\emptyset$, *if* $i\neq j$.

(2) $\Omega(K)=\bigcup_{i=1}^{4}\Omega_i(K)$.

Proof. (1) is easily seen from the above lemmas. To prove that $\Omega(K)$ is contained in $\bigcup_{i=1}^{4}\Omega_i(K)$, take an element (A, B) of $\Omega(K)$ and put $C=A\cap B$. There exists a unique integer i such that $\mathfrak{S}_i(K)-Q_{i+1}(K)$ contains C, $i=1, 2, 3$. Then, considering the star of C in a suitable sub-complex $\mathfrak{S}_j(K)$, we obtain $D_j(A, C)$ and $D_j(B, C)$ and hence we can find the sub-set $\Omega_k(K)$ of $\Omega(K)$ containing the pair (A, B).

Proposition 6. *Let K be an SFM and K_1 be a sub-division of K. And, let (A, B) and (A_1, B_1) are elements of $\Omega_i(K)$ and $\Omega_j(K_1)$, respectively. Putting $C=A\cap B$ and $C_1=A_1\cap B_1$, suppose that $\mathring{A}_1$, $\mathring{B}_1$ and $\mathring{C}_1$ are contained in $\mathring{A}$, $\mathring{B}$ and $\mathring{C}$, respectively. Then, we obtain $i=j$.*

Proof. It is not hard to prove, because, making use of the pseudo radial projection, we can find

$$\dim(D_i(A, C)\cap D_i(B, C))=\dim(D_i(A_1, C_1)\cap D_i(B_1, C_1)) .$$

Now, remember that we assumed $\mathfrak{S}_7(P)=\emptyset$ for any fake manifold P considered in this paper as mentioned in the introduction (for the numbering of the singularities of P, recall Definition 6 and its remark). Then, it is clearly seen that this assumption implies the same condition that $\mathfrak{S}_7(K)$ is empty for any SFM K (in this paper).

In this situation, let us review Proposition 4 as follows.

Proposition 4′. *Let K be a p-SFM with non-empty boundary $\dot{K}$. Then, $\dot{K}$ is a closed $(p-1)$-SFM and we obtain*

$$\mathfrak{S}_i(\dot{K})=\mathfrak{S}_i(K)\cap\dot{K} .$$

for $i=1, 2, 3$.

Proof. It is immediate from the facts that

$$\mathfrak{S}_i(K)\cap\dot{K}=\mathfrak{S}_{i+3}(K) ,$$

$$\mathfrak{S}_{i+3}(K)=\mathfrak{S}_i(\dot{K}) ,$$

for $i=1, 2, 3$.

6. The singular block bundles.

In this section, we make the definition of the singular block bundles and state some basic theorems about them.

Definition 9. (The singular block bundles) Let K be an SFM with dimension k, and let Φ^n the n-dimensional fiber-set. Then, $B_k^n(K)$ is defined to be the set of the polyhedra η satisfying the following three conditions (1), (2) and (3).

(1) For any simplex A of K, there exists a block F_A uniquely, called *the block of* η, whose fiber F is chosen in Φ^n as follows.

$$\begin{cases} F=J^n, \text{ when } A \text{ belongs to } K-\mathfrak{S}_2(K)\,, \\ F=Y^n, \text{ when } A \text{ belongs to } \mathfrak{S}_2(K)-\mathfrak{S}_3(K)\,, \\ F=X^n, \text{ when } A \text{ belongs to } \mathfrak{S}_3(K)\,. \end{cases}$$

(2) η is a polyhedron satisfying

$$\eta=\bigcup_{A\in K} F_A\,,$$

that is, η is the union of the blocks of η, where the simplex A of the SFM K should be identified with $A\times o(F)$ of the block F_A.

(3) (The intersections of the blocks of η)

Let A, B and C be the simplexes of K with $A\cap B=C$, and $(F_1)_A$, $(F_2)_B$ and $(F_3)_C$ the blocks of η over A, B and C, respectively. Then, the intersection

$$(F_1)_A\cap(F_2)_B=((F_1)_A|C)\cap((F_2)_B|C)\,,$$

and there exist strongly proper sub-blocks $(F_1)_C$ and $(F_2)_C$ of $(F_3)_C$ satisfying

$$(F_1)_C=((F_1)_A|C)\,,$$
$$(F_2)_C=((F_2)_B|C)\,.$$

Furthermore, we require the following three conditions from (a) through (c). (See Fig. 2)

(a) When the pair (A, B) is an element of $\Omega_1(K)$, $(F_1)_C\cap(F_2)_C$ should be a trivial sub-block of $(F_3)_C$

(b) When the pair (A, B) is an element of $\Omega_2(K)\cup\Omega_3(K)$, $(F_1)_C$ and $(F_2)_C$ are different as sub-blocks of $(F_3)_C$.

(c) When the pair (A, B) is an element of $\Omega_4(K)$, $(F_1)_C$ and $(F_2)_C$ are the same sub-blocks of $(F_3)_C$.

An element of the set $B_k^n(K)$ is called an (n, k)-*singular block bundle* over the SFM K. Sometimes, if there is no confusion, we call it simply a block bundle or just a bundle.

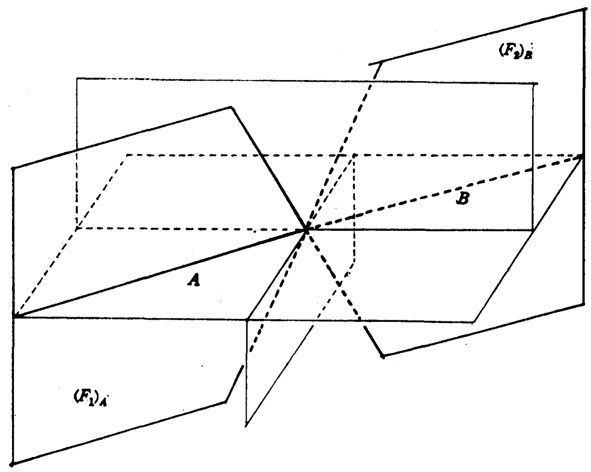

Fig. 2-1.

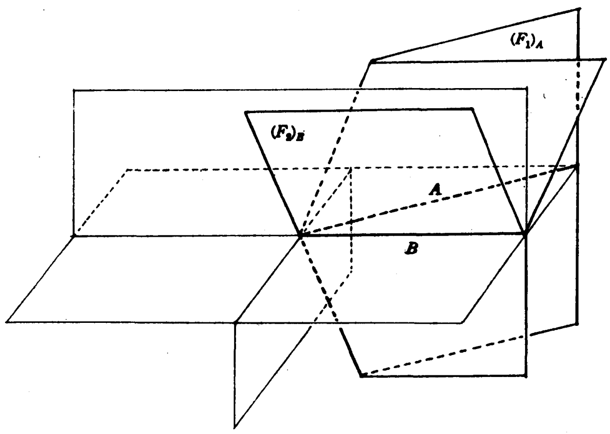

Fig. 2-2.

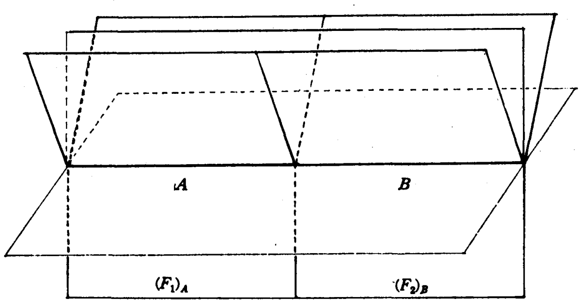

Fig. 2-3.

Now, we define the restricted bundles of an element of $B_k^n(K)$ on the ***subsets*** of the SFM K.

Definition 10. Let L be a sub-set of an SFM K, that is, L is just a set of simplexes of K, and let η be an element of $B_k^n(K)$. Then, the ***restricted bundle***

of η on L is defined by

$$(\eta|L)=\bigcup_{A\in L} F_A ,$$

that is, the restricted bundle $(\eta|L)$ is the union of the blocks F_A of η with A in L.

Remark. In Definition 10, it should be remarked that L is not necessarily an SFM and is not even a sub-complex of K. And, in general, the restricted bundles may not be singular block bundles, even if the sub-set is an SFM.

Now, the isomorphism and the equivalence of the singular block bundles are defined naturally.

Definition 11. Suppose that η_1 and η_2 are two elements of $B_k^n(K)$. Then, we say that η_1 and η_2 are *isomorphic*, denoted by $\eta_1 \approx \eta_2$, whenever there exists a homeomorphism from η_1 onto η_2 which is the identity on K and sends the blocks of η_1 onto those of η_2.

Definition 12. Let K_1 denote a sub-division of an SFM K. Suppose that η and η_1 are elements of B_k^n and $B_k^n(K_1)$, respectively. Then, we say that η_1 is a *sub-division* of η, whenever, for any block F_A of η over a simplex A of K, we obtain

$$F_A=\bigcup_B F_B ,$$

where F_B is the block of η_1 over a simplex B of K_1 such that $\mathring{B}$ is contained in $\mathring{A}$.

Definition 13. Suppose that α and β are elements of $B_k^n(K)$ and $B_k^n(L)$, respectively, and $|K|=|L|$. Then, we say that α and β are *equivalent*, written $\alpha\sim\beta$, whenever there exist sbu-divisions α_1 and β_1 of α and β, respectively, with $\alpha_1\approx\beta_1$.

Kato proved the following in [3].

Theorem A. *Let η be a prebundle over a complex L. If K is collapsible, then η is trivial as a prebundle.*

Applying a similar argument to that he used in [3], we are able to obtain the following propositions.

Proposition 7. *Let η be an (F, o)-prebundle over K with F in Φ^n and K_1 a sub-division of K. Then, there exists a sub-division of η over K_1.*

Corollary to Proposition 7. *Suppose that η is an (F, o)-prebundle over K with F in Φ^n and $|K|$ is collapsible. Then, η is trivial as a prebundle.*

Proof. Assuming Proposition 7, we obtain a proof by the same way as in [3].

Proposition 8. *Let η be an (F, o)-prebundle over K with F in Φ^n and K_1 a sub-division of K. Suppose that η_1 and η_2 are two sub-divisions of η over K_1. Then, there exists an isomorphism between η_1 and η_2 which is isotopic to the identity as a homeomorphism of η onto itself keeping K fixed.*

Proof. The proof of Proposition 7 (Corollary to Proposition 7) and Proposition 8 almost parallels that of [3] or [4]. It goes by induction of $k=\dim K$. If $k=0$, there is nothing to prove. And, if $F=J^n$, they are already proved in [3] and [4]. So, we may assume that F is either Y^n or X^n. First of all, note the following statement (*).

(*) *Suppose that A is a simplex and h is a homeomorphism of $F\times A$ onto itself. Then, we have*

$$h(\text{core}(F)\times A)=\text{core}(F)\times A .$$

A proof of (*) may be obtained easily. Let us consider the connected components E_i' of $F\times A-\text{core}(F)\times A$. Then, the closure E_i of E_i' is a ball with a common face $\text{core}(F)\times A$. Furthermore, we can regard E_i as $\text{core}(F)\times A\times I$, because F is either $\text{core}(Y^n)\times Y^1$ or $\text{core}(X^n)\times X^1$ and I is chosen to be a semi-proper sub-fiber of Y^1 or X^1. We show that any homeomorphism f from $F\times \dot{A}$ onto itself can be extended to a homeomorphism g from $F\times A$ onto itself which is isotopic to the identity keeping A fixed provided that f is isotopic to the indentity keeping $\dot{A}$ fixed. Let us consider the restriction f_1 of f on $\text{core}(F)\times \dot{A}$. Since f_1 is isotopic to the identity keeping $\dot{A}$ fixed and $\text{core}(F)\times A$ is a (J^{n-1}, o)-prebundle over A, we can extend f_1 to a homeomorphism of $\text{core}(F)\times A$ onto itself which is isotopic to identity keeping A fixed. This extension is written f_2. Then, f_2 can be extended to a homeomorphism f_{i3} of E_i onto itself which is isotopic to the identity, because $F\times \dot{A}\cup \text{core}(F)\times A$ is an $(n+k-1)$-face of the $(n+k)$-ball E_i. Now, combining f_{i3}, we obtain the required homeomorphism g. And, the required isotopy is also obtained, because the isotopies in the above extension steps can be chosen to be the extensions of the formers. Then. the rest of the proof is not hard to see, using the skeleton-wise extension argument.

From the above propositions, we obtain the existence and uniqueness of sub-division of singular block bundles as follows.

Proposition 9. (a) *Let η be an element of $B_k^n(K)$ and K_1 a sub-division of K. Then, there exists a sub-division of η in $B_k^n(K_1)$.*

(b) *Let η_1 and η_2 be elements of $B_k^n(K_1)$. If η_1 and η_2 are sub-divisions of η, then there exists an isomorphism between η_1 and η_2 which is isotopic to the* *identity keeping K fixed.*

Proof. Let A be a simplex of K. Then, it is known that the restricted bundle $(\eta|A)$ of η on A can be regarded as an (F, o)-preblundle by the natural structure defined below. Note that $(\eta|A)$ is the block F_A of η over A. Let B denote a proper face of A. We define the block F_B over B to be the sub-block of the block G_B of η over B which appears as the intersection of F_A and G_B. Thus, we can define the blocks over all the faces of A. It is clear that they together with F_A make $(\eta|A)$ an (F, o)-prebundle over $\bar{A}$ with F in Φ^n, where $\bar{A}$ means the simplicial complex consisting of A and its faces. We write this prebundle $F(\bar{A})$. Let $A_1, \cdots, A_n$ be the simplexes of K arranged in the order of increasing dimension. Then, applyng Proposition 7 and Proposition 8 to $F(\bar{A}_i)$, we have the required properties by induction.

Since the isomorphisms and the equivalences of (n, k)-singular block bundles are equivalence relations, we regard $B_k^n(K)$ and $B_k^n(|K|)$ as the sets of the isomorphism classes and the equivalence classes of (n, k)-singular block bundles over K, respectively. When we write $B_p^n(P)$ for a fake p-manifold P, it always means the set of the equivalence classes of the (n, p)-singular block bundles over an SFM K with $P=|K|$.

Now, we easily obtain the following.

Proposition 10. *The correspondence from $B_k^n(K)$ to $B_k^n(|K|)$ defined by sending each (n, k)-singular block bundles to its equivalence class is a bijection.*

Here, we introduce the notion of sub-SFM in SFM.

Definition 14. Let L be a sub-complex of an SFM K. Then, L is said to be a sub-SFM of K, when L is an SFM and $\mathfrak{S}_i(L)-\mathfrak{S}_{i+1}(L)$ is contained in $\mathfrak{S}_i(K)-\mathfrak{S}_{i+1}(K)$ for $i=1, 2, 3$. (See Fig. 3).

We have a relation between our singular block bundles and the combnatorial prebundles or the block bundles.

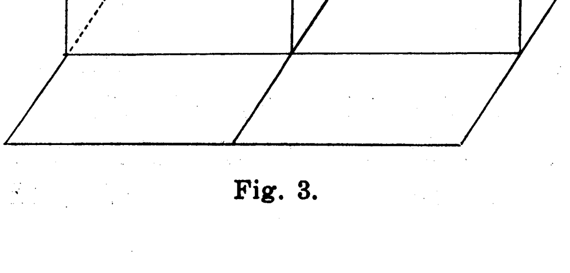

Fig. 3.

Theorem 1. *Let η be an element of $B_k^n(K)$. If $\mathfrak{S}_2(K)$ is empty, then η is a combinatorial prebundle or a block bundle over K.*

Proof. We can prove it directly and easily by comparing their definitions.

It is clearly seen that there exists a singular block bundle which is neither a block bundle nor a prebundle. Conversely, a block bundle is a singular block bundle. But there exists a combinatorial prebundle which is not a singular block bundle.

For an element η of $B_k^n(K)$, we write the *boundary of the singular block bundle* η as $\dot{\eta}$ and the *boundary of the polyhedron* η as $(\eta)^{\cdot}$.

Suppose that a polyhedron P collapses to a proper sub-polyhedron Q of P. Put the collapsing α. We say that α is *admissible* if α is obtained by a collapsing α_1 followed by a second one α_2 such that α_1 is a collapsing of P to $Q \cup N(\dot{Q}, \dot{P})$ and α_2 is one of $Q \cup N(\dot{Q}, \dot{P})$ to Q, where $N(\dot{Q}, \dot{P})$ denotes a regular neighborhood of $\dot{Q}$ in $\dot{P}$.

Lemma 10. *Let η be an element of $B_k^n(K)$. Suppose that (A, B) is an element of $\Omega_2(K)$ and $C=A\cap B$. Let $(F_1)_A$, $(F_2)_B$ and $(F_3)_C$ are the blocks of η over A, B and C, respectively.*

(1) *If A and B are contained in $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$, then $(F_1)_A\cap(F_2)_B$ is a strongly proper sub-block of $(F_3)_C$.*

(2) *If either A or B is not in $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$, then, $(F_1)_A\cap(F_2)_B$ is a semi-proper sub-block of $(F_3)_C$.*

Proof. (1) We may regard F_1 and F_2 as Y^n, because both A and B are contained in $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$. And, by Lemma 9, C is contained in $\mathfrak{S}_3(K)$. Hence $F_3=X^n$. From the definition, F_1 and F_2 are different in F_3. Then, it is clear that $F_1\cap F_2=J^n$, and hence we see that

$$(F_1)_A\cap(F_2)_B=(J^n)_C\ ,$$

is a strongly proper sub-block of $(X^n)_C=(F_3)_C$.

(2) By the similar argument to the above, we obtain the result.

Lemma 11. *Let η be an element of $B_k^n(K)$. Suppose that (A, B) is an element of $\Omega_3(K)$ and $C=A\cap B$. Let $(F_1)_A$, $(F_2)_B$ and $(F_3)_C$ are the blocks of η over A, B and C, respectively. Then, either $(F_1)_C=((F_1)_A|C)$ or $(F_2)_C=((F_2)_B|C)$ equals to $(F_3)_C$.*

Proof. It is not hard to prove by the similar argument to one used in the proof of Lemma 10.

Here, we state a theorem which is essentially important in the singular block bundle theory. It shows the difference between the combinatorial prebundles and the singular block bundles.

Theorem 2. *Let η be an element of $B_k^n(K)$.*

(1) *η is an $(n+k)$-manifold in which K is properly embedded.*

(2) $(\eta)^{\cdot}=(\eta|\dot{K})\cup\bigcup_{A\in K}\dot{F}_A$, *where F_A is the block of η over A.*

(3) *η collapses to K admissibly.*

Proof. To show (1), it is sufficient to prove that $(\eta|st(v,K)-lk(v,K))=A$ is an $(n+k)$-ball.

Case 1. Suppose that v is a vertex contained in $K-\mathfrak{S}_2(K)$. Then, $st(v,K)$ is a k-ball and hence it is easily seen that A is an $(n+k)$-ball.

Case 2. Suppose that v is a vertex contained in $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$. Let E_1, E_2 and E_3 denote the closures of the connected components of

$$|st(v,K)|-|st(v,\mathfrak{S}_2(K))|\ .$$

Then, E_i can be regarded as a set of simplexes of $K-\mathfrak{S}_2(K)$ and is a k-ball. Thus, $(\eta|E_i)$ is an $(n+k)$-ball for $i=1,2,3$. On the other hand, $(\eta|st(v,\mathfrak{S}_2(K))-lk(v,\mathfrak{S}_2(K)))$ is homeomorphic to $Y^n\times st(v,\mathfrak{S}_2(K))$. And, $Y^n\times st(v,\mathfrak{S}_2(K))$ can be written as $Y^n\times B^{k-1}$, since $st(v,\mathfrak{S}_2(K))$ is a $(k-1)$-ball B^{k-1}. Now, $Y^n\times B^{k-1}$ is disconnected into three components E_1', E_2' and E_3' where E_i' is an $(n+k-1)$-ball with $E_i'\cap E_j'=\text{core}(Y^n)\times B^{k-1}$, that is, E_1', E_2' and E_3' are the closures of the connected components of $Y^n\times B^{k-1}-\text{core}(Y^n)\times B^{k-1}=(Y^n-\text{core}(Y^n))\times B^{k-1}$. Now, put $(\eta|E_1)\cap(\eta|E_2)=(\eta|E_1)^{\cdot}\cap(\eta|E_2)^{\cdot}=E_1'$. It is seen from the definition. Hence, it is known that $A'=(\eta|E_1)\cup(\eta|E_2)$ is an $(n+k-1)$-ball. And, then, $A'\cap(\eta|E_3)=\dot{A}'\cap(\eta|E_3)^{\cdot}=E_2'\cup E_3'$ again from the definition. Since $A=A'\cup(\eta|E_3)$, A must be an $(n+k)$-ball.

Case 3. Suppose that v is contained in $\mathfrak{S}_3(K)$.

Step 1. In this case, there exist six connected components in

$$|st(v,K)|-|st(v,\mathfrak{S}_2(K))|\ ,$$

each of whose closures is written E_i, $i=0,\cdots,5$, and the numbering of E_i's is chosen so that

$$\begin{cases}\dim(E_i\cap E_{i+j})=k-1, & j=1,2,\\ \dim(E_i\cap E_{i+3})=k-2\ ,\end{cases}$$

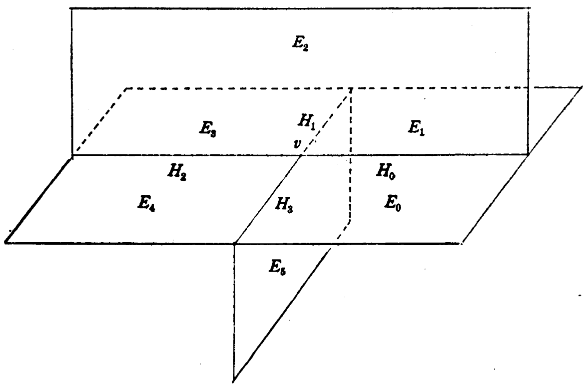

Fig. 4.

where the numbers are taken by mod 6, (See Fig. 4). Then, E_i is a k-ball for $i=0, \cdots, 5$, and E_i can be regarded as a set of simplexes of $K-\mathfrak{S}_2(K)$. And, it is known that $(\eta|E_i)$ is an $(n+k)$-ball for $i=0, \cdots, 5$.

Step 2. Let us consider

$$P=|st(v, \mathfrak{S}_2(K))|-|st(v, \mathfrak{S}_3(K))| .$$

Then, P has four connected components and their closures H_i, $i=0, \cdots, 3$, are $(k-1)$-balls satisfying

$$H_i \cap H_j=\dot{H}_i \cap \dot{H}_j=st(v, \mathfrak{S}_3(K)) ,$$

where $i \neq j$ and of course $st(v, \mathfrak{S}_3(K))$ is a $(k-2)$-ball. And, H_i can be regarded as a set of simplexes contained in $\mathfrak{S}_2(K)-\mathfrak{S}_3(K)$ and hence $(\eta|H_i)$ is $Y^n \times H_i$. We may assume the numbering of H_i's as follows. (See Fig. 4)

$$\begin{cases} E_0 \cap E_1=H_0 , & E_1 \cap E_3=H_1 , \\ E_3 \cap E_4=H_2 , & E_4 \cap E_1=H_3 . \end{cases}$$

Now E'_{ij}, $j=1. 2, 3$, is defined to be the closure of a connected component of $(Y^n-\text{core}(Y^n)) \times H_i$. Of course, E'_{ij} is an $(n+k-1)$-balls.

Step 3. Put $A_i=(\eta|E_0) \cup \cdots \cup (\eta|E_i)$. First, it is known that A_1 is an $(n+k)$-ball, because both $(\eta|E_0)$ and $(\eta|E_1)$ are $(n+k)$-balls as mentioned in Step 1 and $(\eta|E_0) \cap (\eta|E_1)=(\eta|E_0)^{\cdot} \cap (\eta|E_1)^{\cdot}$ is some E'_{0j}, say E'_{01}. Then, $A_1 \cap (\eta|E_2)$ is a common face $E'_{02} \cup E'_{03}$ of A_1 and $(\eta|E_2)$. Thus, A_2 is an $(n+k)$-ball. Now, we prove that A_3 is an $(n+k)$-ball. We may have

$$(\eta|E_2) \cap (\eta|E_3)=E'_{21} ,$$

$$(\eta|E_1) \cap (\eta|E_3)=E'_{11} .$$

Note that $E'_{21}\cup E'_{11}$ is an $(n+k-1)$-ball because $E'_{21}\cap E'_{11}=\dot{E}'_{21}\cap\dot{E}'_{11}$ is an $(n+k-2)$-ball which is the closure of a connected component of $(X^n-\text{core}(X^n))\times st(v, \mathfrak{S}_3(K))$. Since $A_2\cap(\eta|E_3)=\dot{A}_2\cap(\eta|E_3)^{\cdot}=E'_{21}\cup E'_{11}$ is a common face of A_2 and $(\eta|E_3)$, A_3 must be an $(n+k)$-ball. Using the similar argument to the above, successively, it is known that A_4 and A_5 $(=A)$ are $(n+k)$-balls which is the required property.

(2) It is rather trivial.

(3) Note that $F\times A$ collapses to $F\times\dot{A}$ where F is a collapsible polyhedron and A is a simplex. Suppose that $A_1, \cdots, A_m$ be the simplexes of $K-\dot{K}$ arranged in the order of decreasing dimension. Since our fibers J^n, Y^n and X^n are collapsible, we can apply the above collapsing to the block $(F_i)_{A_i}$ of η, inductively. Then, we obtain the required admissible collapsing from η to K.

Theorem 3. *Let L be a sub-SFM of an SFM K and η an element of $B^n_k(K)$. Then, $(\eta|L)$ is an element of $B^n_p(L)$, where $p=\dim L$.*

Proof. It is immediate from the definition.

7. Regular neighborhoods of locally unknotted fake manifolds in manifolds.

In Theorem 2, it is shown that any element of $B^n_p(P)$ is an abstract regular neighborhood of P. In this section, we consider about regular neighborhoods of fake manifolds in manifolds, as a converse to the above.

First of all, let us introduce the concept of "local unknottedness" of fake manifolds in manifolds.

Definition 14. Let St be an element of $\mathscr{S}^p$ and let B denote the q-ball defined by

$$B=\{(x_1, \cdots, x_q)\,|\,|x_i|\leqq 1\}\,.$$

Then, the pair (B, St) is colled a ***standard pair***.

Definition 15. Let P be a fake p-manifold properly embedded in a q-manifold V. Take a point x of P. We say that P is ***locally unknotted at x in V***, if the pair $(st(x, V), st(x, P))$ is homeomorphic to the standard pair. And, if P is locally unknotted at any point of P in V, we say that P is ***locally unknotted in V***.

The purpose of this section is to show the following theorem.

Theorem 4. *Let P be a locally unknotted fake p-manifold in a q-manifold V and N the regular neighborhood of P in V meeting the boundary regularly. Then, N belongs to $B^{q-p}_p(P)$ and $(N|\dot{P})=N\cap\dot{V}$.*

Now, we state some lemmas.

Lemma 12. *Let (B, St) be a standard pair. Then, B belongs to $B_p^{q-p}(St)$.*

Proof. *Case 1.* When $St=St_1$, we can regard $B=St\times J^{q-p}$. Hence, B is an element of $B_p^{q-p}(St)$.

Case 2. Suppose $St=St_2$. The proof of this case is done through two steps.

Step 1. First, we prove the case when $q-p=1$. Let us consider $B-St$ which has three connected components each of whose closures is a q-ball, denoted by B_1, B_2 and B_3, with $St\cap B_i=A_i$ a $(q-1)$-face of B_i, $i=1, 2, 3$. We can regard $B_i=A_i\times[0, 1]$ with $A_i=A_i\times 0$. Since $\mathfrak{S}_2(St)$ is a $(p-1)$-ball contained in A_i, $\mathfrak{S}_2(St)\times[0, 1]=C_i$ is a proper p-ball in B_i. Put $C=C_1\cup C_2\cup C_3$. Then, it is clear to see $C=\mathfrak{S}_2(St)\times Y$. C is the union of the blocks over $\mathfrak{S}_2(St)$. Now, $B-C$ has three connected components each of whose closures is a q-ball, denoted by D_1, D_2 and D_3. It is seen that $C\cap D_i$ is a $(q-1)$-face of D_i which is a union of sub-blocks of C. Put $S_i=St\cap D_i$. Then, it is not hard to see $D_i=S_i\times J$, where J is a strongly proper sub-fiber of Y. Thus, B is an element of $B_p^{q-p}(St)$.

Step 2. Here, we deal with the case $q-p\geqq 2$. Let us consider the $(p+1)$-ball B' defined by

$$B'=\{(x_1, \cdots, x_{p+1}, 0, \cdots, 0)\,|\,|x_i|\leqq 1\}\,.$$

Then, (B', St) is a standard pair with codimension 1 and we can write $B=B'\times J^{q-p-1}$. Since B' is a singular block bundle over St by the structure obtained in Step 1, it is easily seen that B is an element of $B_p^{q-p}(St)$ by taking the fibers to be $F\times J^{q-p-1}$, where F means the fiber of B'.

Case 3. Suppose $St=St_3$. By a similar argument to that of Case 2, we obtain the conclusion.

Proof of Theorem 4. By the similar argument to that of [4], together with Lemma 12 above, it is not hard to obtain a proof of Theorem 4.

Finally, we state a theorem about 3-manifolds. Note that any 3-manifold with boundary has a closed fake surface as its spine [1]. And, it is clear that any fake surface properly embedded in a 3-manifold is locally unknotted. Thus, we have the following.

Theorem 5. *Let V be a 3-manifold with boundary. Then, there exists a closed fake surface P such that V is a singular block bundle over P with fiber-set Φ^1.*

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