PROXIMAL POINTS OF CONVEX SETS IN NORMED LINEAR SPACES*

By

D. V. PAI

(Received February 9, 1973)

1. Introduction

Let U, V be two non-empty convex sets in a normed linear space X. We call the points $\bar{u} \in U$, $\bar{v} \in V$ the proximal points of the sets U, V if and only if $||\bar{u}-\bar{v}|| \leq ||u-v||$ for all points $u \in U$ and $v \in V$. In this paper, we are interested to study the existence, characterizations and the uniqueness of proximal points and also to obtain some duality results for $d(U, V)$, the distance between the two convex sets. Our principal aim, however, is to obtain characterizations of proximal points. For this purpose we immediately note that the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if $\bar{u}-\bar{v}$ is a point of the minimum norm (projection point of 0) in the convex set $U-V=\{u-v/u \in U, v \in V\}$. This observation would easily enable one to obtain information about the proximal points $\bar{u}$, $\bar{v}$ of U, V in terms of suitable conditions on the set $U-V$. However, for theoretical as well as practical purposes, it seems more meaningful and conveninet to formulate information about the proximal points in terms of conditions on the individual sets U, V, rather than in terms of conditions on the set $U-V$. In the present exposition we mainly attempt to obtain such results. This approach gives a unified theory, which yields as special cases, most of the well known results concerning best approximation from the elements of a convex set due to Garkavi [7] and [8], Havinson [9], Nikolskii [12], Deutsch and Maserick [4] etc., when one of the two convex sets U, V is reduced to a single point. The techniques employed in proving most of the results in this paper are essentially geometric in nature.

Section 2 gives the definitions and basic notations to be used in the rest of the paper. In Section 3, we develop the geometric tools to be used elsewhere. The principal result of this section is Theorem 3.2 which extends for the case of

* Abstract of this paper has appeared in the Notices of the A. M. S. (Abstract 71T-B 205) Oct. 1971. This paper was presented by the author at 'The Fourth MASTECH Conference on Functional Analysis and its applications to Science and Technology' held at Bangalore, India in September 1971.

two convex sets, the separation principle of a point and a convex set employed by Garkavi [7]. In Section 4, we conside characterizations of proximal points. Theorem 4.4 generalizes for proximal points a corresponding result of Deutsch and Maserick [4] (also reproved by Havinson [9]) for best approximation (projection point) from the elements of a convex set. Next we consider the case when the dual space X^* is strictly convex and prove that in this case the points $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if $\bar{u}\in U$ is the projection point of $\bar{v}$ and $\bar{v}\in V$ is the projection point of $\bar{u}$ (Theorem 4.8). This generalizes, for arbitrary convex sets, a well known result of Cheney and Goldstein [2] for the case when X is a Hilbert space and U, V are closed convex sets. A practical motivation for this result is the following more general convex minimization problem which is often important in concrete situations.

Suppose U, V are disjoint convex sets in a normed linear space X and Φ: $U\times V\to \boldsymbol{R}^+$ is a functional, which is convex in each individual variable u, v (the other variable being fixed). Then under what conditions, for fixed $\bar{u}, \bar{v}$, the minimization of Φ in each individual variable ensures the minimization of Φ in both the variables? Theorem 4.8 answers this question for the special case $\Phi(u, v)=||u-v||$. Theorem 4.10 combines (and generalizes) the well known characterizations of best approximation from the elements of a linear subspace (Singer [13]) and from the elements of a convex set (Deutsch and Maserich [4], also Havinson [9]). Theorems 4.12 and 4.14 give the applications of Theorem 4.10 to the spaces $L_p(E, \Sigma, \mu)$ ((E, Σ, μ) being a σ-finite measure space) for the cases $p=1$ and $1<p<\infty$ respectively.

In Section 5, we consider some additional characterizations of proximal points using extreme functionals. Firstly, we restrict our consideration to the case when both the convex sets U, V are contained in a finite dimensional subspace of X and show that in this case Theorem 4.4 (and hence the majority of the results of Section 4 which depend on this theorem) can be significantly improved (Theorem 5.4). We next introduce the notion of $(\bar{u}, \bar{v})$-symmetry of convex sets U, V ($\bar{u}\in U$, $\bar{v}\in V$ being a fixed pair of points). The assumption of the hypothesis that the sets U, V are $(\bar{u}, \bar{v})$-symmetric enables us to obtain a Garkavi-type characterization (cf. [8]) for proximal points $\bar{u}, \bar{v}$ in terms of the extreme points of the unit sphere of X^* (Theorem 5.10). For the special case when $X=\mathscr{C}(T)$ (T compact Hausdorff) (resp. X is a subspace of $\mathscr{C}(T)$ with ihe induced norm), we use the well known representations for the extreme points of the unit ball of X^* and Theorem 5.10 to obtain a Kolmogorov-type characterization (cf. [10]) for proximal points (Theorem 5.11 resp. Theorem 5.13).

In Section 6, we introduce (following Efimov and Stečkin [6]) the notions of semi-Chebyshev and Chebyshev pair of convex sets and obtain as easy consequences of Theorem 4.4, the necessary and sufficient conditions for the pair U, V to be Chebyshev and semi-Chebyshev respectively (Theorems 6.1, 6.2 respectively).

In Section 7, we are mainly concerned with the problem of estimating $d(U, V)$ by employing some duality results. Theorem 7.1 expresses this as a maximization problem in the dual space X^*. On the other hand, if the two convex sets are in X^*, Theorem 7.4 reduces this problem to an equivalent maximization problem in X. Theorem 7.7 combines these two results to some extent in a dual manner. Theorems 7.1, 7.4 and 7.7 generalize the corresponding duality results of Garkavi [7].

It is a pleasure to thank Dr. P. C. Jain, Professor in the Department of Mathematics, Indian Institute of Technology, Bombay, for encouragement and many helpful discussions.

2. Definitions and Notations

Throughout in this and the following sections, let X be a normed linear space (real or complex) with norm $\|\cdot\|$ and let X^* denote the normed dual of X, i.e., the space of continuous linear functionals $L: X \to \boldsymbol{R}$ (or $\boldsymbol{C}$) normed by the usual operator norm $\|L\| = \sup_{\|x\|=1} |L(x)|$. We denote by S_{X^*} the unit sphere $\{L \in X^*/\|L\|=1\}$ and by K_{X^*} the closed unit ball $\{L \in X^*/\|L\| \leq 1\}$ of X^*. For each $x \in X$, let $\hat{x}$ denote the cannonical image of x in X^{**}, i.e. the functional defined by $\hat{x}(x^*) = x^*(x)$, $x^* \in X^*$. If $E \subseteq X$, then let $\hat{E}$ denote the set $\{\hat{x}/x \in E\}$. By the $\sigma(X^*, X)$-topology on X^* we will understand the weakest linear topology which can be imposed on X^* such that each $\hat{x} \in \hat{X}$ is continuous. We call a set $E \subseteq M$ an extremal subset of M if a proper convex combination $\alpha x_1 + (1-\alpha)x_2$, $0<\alpha<1$ of two points $x_1, x_2 \in M$ is in E only if both x_1 and x_2 are in E. An extremal subset of M consisting of just one point is called an extreme point of M. We shall denote by $\mathscr{E}(M)$ the set of all the extreme points of M. A normed linear space X is called strictly convex if $S_X = \mathscr{E}(S_X)$ and it is called uniformly convex if for each $\varepsilon > 0$, there exists a $\delta > 0$ such that $\|x\| = \|y\| = 1$ and $\|(x+y)/2\| > 1-\delta$ imply $\|x-y\| < \varepsilon$.

The distance $d(U, V)$ between the two sets is given by $\inf_{u \in U, v \in V} \|u-v\|$. The points $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if $\|\bar{u}-\bar{v}\| = d(U, V)$. Unless stated otherwise all of our other notations will conform to those given in [5].

3. Existence of Proximal Points and a Basic Separation Principle

If the convex sets U, V are both compact, then proximal points of U and V exist. This follows immediately from the continuity of the function $||u-v||$ on $U\times V$ and the compactness of $U\times V$. In general, if U and V are both closed and one of them is compact, the proximal points may not exist for an arbitaray normed linear space X. However, if X is a uniformly convex Banach space, the proximal points exist for this case. This is given by the following theorem.

Theorem 3.1. Let X be a uniformly convex Banach space and U, V be two closed convex sets such that one of them is compact. Then the proximal points $\bar{u}, \bar{v}$ of U and V exist and they are unique if $(U-U)\cap(V-V)=\{0\}$.

Proof. $U-V$ is convex if both U and V are convex and $U-V$ is closed if both U and V are closed and one of them is compact. Thus $U-V$ is a closed convex set. Since X is a uniformly convex Banach space, by a well known result [cf. [1], pp. 22] $U-V$ contains a point of the minimum norm. Hence, there exist $\bar{u}\in U$, $\bar{v}\in V$ such that $||\bar{u}-\bar{v}||\leq||u-v||$ for all points $u\in U$ and $v\in V$. This establishes the existence of proximal points. To prove the uniqueness, suppose $\tilde{u}, \tilde{v}$ are proximal points of U and V, distinct from $\bar{u}, \bar{v}$. Then $||\bar{u}-\bar{v}||=||\tilde{u}-\tilde{v}||=d(U, V)$. This gives $||(\bar{u}+\tilde{u})/2-(\bar{v}+\tilde{v})/2||=d(U, V)$. Since a uniformly convex normed space is also strictly convex, we get $\bar{u}-\bar{v}=\tilde{u}-\tilde{v}$ i.e. $\bar{u}-\tilde{u}=\bar{v}-\tilde{v}$ and by the condition $(U-U)\cap(V-V)=\{0\}$, $\bar{u}=\tilde{u}$ and $\bar{v}=\tilde{v}$.

Remark. If one of U and V say U is a linear subspace, then the condition $(U-U)\cap(V-V)=\{0\}$ is equivalent to $U\cap(V-V)=\{0\}$.

In what follows we shall not make the additional assumptions of Theorem 3.1 on the convex sets U, V in order to ensure the existence of proximal points. However, many times we shall assume that the proximal points exist. For practical purposes, this is often a reasonably good assumption to start with for studying the characterizations of proximal points. In the following, we prove a geometric result which is essentially a consequence of the separation form of the Hahn-Banach theorem. This will be our main tool for discussing the characterizations of proximal points in the next section. Incidentally, this also extends for the case of two convex sets, the separation principle of a point and a convex set employed by Garkavi [6] for studying duality results for approximations from convex sets (also see Deutsch and Maserick [4]).

Theorem 3.2. Let U, V be convex subsets of X, such that $d(U, V)>0$. Then there exists an $L\in X^*$ such that

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} \quad L \in S_{X^*} \text{ and} \\ &\text{(ii)} \quad d(U, V) = \inf Re\, L(U) - \sup Re\, L(V)\,. \end{aligned}$$

Proof. We need the following lemmas:

Lemma 3.3. Let K be a non-empty set in X and for $0 \neq L \in X^*$, H denote the hyperplane

(3.2) $$\{x \in X / Re\, L(x) = a\}, \text{ then}$$

(3.3) $$d(K, H) = \|L\|^{-1} \inf_{k \in K} |Re\, L(k) - a|\,.$$

From a well-known result of Ascoli (cf. [15], pp. 24), it follows that for a fixed $k \in K$, $d(k, H) = \|L\|^{-1}\ |Re\, L(k) - a|$. Hence

$$d(K, H) = \inf_{k \in K} d(k, H) = \|L\|^{-1} \inf_{k \in K} |Re\, L(k) - a|\,.$$

Lemma. 3.4. Let H be the hyperplane defined by (3.2), then

(1) If H separates the two convex sets U, V in X, then

(3.4) $$d(U, H) \leq d(U, V) \quad \text{and} \quad d(V, H) \leq d(U, V)\,.$$

(2) In each neighbourhood of a point in H, there exist points strictly on either side of the hyperplane i. e. points h_1, h_2 such that

(3.5) $$Re\, L(h_1) < a < Re\, L(h_2)\,.$$

The lemma is geometrically obvious and can be easily verified. We omit the details. To prove the theorem, put $S = \{x \in X / d(x, U) < d(U, V)\}$. Then S is convex and every point of U is an interior point of S. Also $S \cap V = \Phi$. By the Hahn-Banach theorem in separation form (due to Tukey cf. [5] pp. 417), there exists an $L \in X^*$ such that $\|L\| = 1$ and

(3.6) $$\sup Re\, L(V) \leq \inf Re\, L(S) \leq \inf Re\, L(U)\,.$$

(Here and in other places, we abbreviate $\inf_{s \in S} Re\, L(s)$ by $\inf Re\, L(S)$ and a similar abbreviation for the supremum). Let H denote the hyperplane

(3.7) $$\{x \in X / Re\, L(x) = \sup Re\, L(V)\}, \text{ then } H \text{ separates } U \text{ and } V.$$

Using Lemmas 3.3 and 3.4

$$\inf Re\, L(U) - \sup Re\, L(V) = d(U, H) \leq d(U, V)\,.$$

If we assume that $d(U, H) < d(U, V)$, then there exists an $h \in H$ such that $d(U, h) < d(U, V)$. Hence $h \in S$ and S is a neighbourhood of h which lies on one side of H. This contradicts Lemma 3.4 (2), giving $d(U, H) = d(U, V)$ and the proof is complete.

Theorem 3.2 states a geometrically evident fact that the distance $d(U, V)$ between the two convex sets can be expressed as the distance between one of the two sets and a suitable hyperplane or even as the distance between two parallel hyperplanes, namely the hyperplane H given by (3.7) and the hyperplane $\tilde{H}$ given by

$$\tilde{H}=\{x \in X / Re\, L(x)=\inf Re\, L(U)\}\,. \tag{3.8}$$

Moreover, if in this case one assumes that the proximal points $\bar{u} \in U$ and $\bar{v} \in V$ exist, then the conclusion of Theorem 3.2 can be strengthened. In this case the separating hyperplanes H and $\tilde{H}$ given by (3.7) and (3.8) support V, U at $\bar{v}, \bar{u}$ respectively, i. e.,

$$Re\, L(\bar{u})=\inf Re\, L(U) \quad \text{and} \quad Re\, L(\bar{v})=\sup Re\, L(V)\,.$$

This is given by the following corollary:

Corollary 3.3. Let U, V and H be as in the last theorem. If $\bar{u} \in U$, $\bar{v} \in V$ are proximal points then

$$\begin{aligned} &Re\, L(\bar{u})=\inf Re\, L(U) \quad \text{and} \\ &Re\, L(\bar{v})=\sup Re\, L(V)\,. \end{aligned} \tag{3.9}$$

Proof. Since $d(\bar{v}, U)=d(U, V)$, $\bar{v} \in cl(S)$. (Here and in other places $cl(S)$ will denote the closure of S).

Hence from (3.6) $Re\, L(\bar{v}) \geq \inf Re\, L(cl(S))=\inf Re\, L(S) \geq \sup Re\, L(V)$

Also $Re\, L(\bar{v}) \leq \sup Re\, L(V)$. Thus $Re\, L(\bar{v})=\sup Re\, L(V)$.

Next from (3.1)

$$\begin{aligned} \|\bar{u}-\bar{v}\| &=\inf Re\, L(U)-\sup Re\, L(V) \\ &=\inf Re\, L(U)-Re\, L(\bar{v}) \\ &\leq Re\, L(\bar{u})-Re\, L(\bar{v})=Re\, L(\bar{u}-\bar{v}) \\ &\leq |L(\bar{u}-\bar{v})| \\ &\leq \|\bar{u}-\bar{v}\|\,. \end{aligned}$$

Hence equality must prevail throughout this chain of inequalities, giving $Re\, L(\bar{u}) =\inf Re\, L(U)$. This proves the corollary.

4. Characterizations of Proximal Points

In this section, we obtain characterizations of proximal points which are essentially geometric in character. Firstly we note that the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if $\bar{u}-\bar{v} \in U-V$ is a point of the minimum norm (projection point of 0 in $U-V$). Hence, we can at once apply the well known characterizations of best approximations from the elements of convex sets and obtain the following:

Theorem 4.1. (*Singer*) Let U, V be convex subsets of X such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if, there exists $L\in S_{X^*}$ such that

$$Re\, L(u-v)\geq||\bar{u}-\bar{v}|| \quad (u\in U, v\in V)\,. \tag{4.1}$$

Theorem 4.2. (*Deutsch and Maserick, Havinson*) Let U, V be convex subsets of X such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if there exists an $L\in X^*$ such that

$$\begin{aligned} &(\mathrm{i})\ \ L\in S_{X^*}\,, \\ &(\mathrm{ii})\ \ Re\, L(\bar{u}-\bar{v})\geq Re\, L(u-v), \quad (u\in U,\ v\in V)\,, \\ &(\mathrm{iii})\ \ L(\bar{u}-\bar{v})=||\bar{u}-\bar{v}||\,. \end{aligned} \tag{4.2}$$

Theorem 4.3. (*Garkavi*) Let U, V be convex subsets of X such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if for each $(u, v)\in U\times V$, there exists an $L=L_{(u,v)}\in X^*$ such that

$$\begin{aligned} &(\mathrm{i})\ \ L\in \mathscr{E}(S_{X^*})\,, \\ &(\mathrm{ii})\ \ Re\, L(\bar{u}-\bar{v})\geq Re\, L(u-v)\,, \\ &(\mathrm{iii})\ \ L(\bar{u}-\bar{v})=||\bar{u}-\bar{v}||\,. \end{aligned} \tag{4.3}$$

In the above theorems, the characterizations of proximal points are essentially expressed in terms of conditions on the set $U-V$. However, for practical purposes, it is often convenient to have characterizations in terms of conditions on the individual sets U, V. In the following, we propose to obtain characterizations of this type. Here we shall mainly employ Theorem 3.2 and Corollary 3.3.

Theorem 4.4. Let X be an arbitrary normed linear space and U, V be convex subsets of X, such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if there exists an $L\in X^*$ such that

$$\begin{aligned} &(\mathrm{i})\ \ L\in S_{X^*}, \\ &(\mathrm{ii})\ \ Re\, L(\bar{u}-u)\leq 0\,, \quad \text{for each } u\in U\,. \\ &\quad \text{and } Re\, L(\bar{v}-v)\geq 0\,, \quad \text{for each } v\in V\,, \\ &(\mathrm{iii})\ \ L(\bar{u}-\bar{v})=||\bar{u}-\bar{v}||\,. \end{aligned} \tag{4.4}$$

Proof. The 'necessity' part follows from Corollary 3.3. To prove the 'sufficiency' part, let $L \in X^*$ satisfy (i), (ii) and (iii). Then for each $u \in U$ and $v \in V$, we have

$$\begin{aligned}||\bar{u}-\bar{v}|| &= L(\bar{u}-\bar{v}) = Re\, L(\bar{u}) - Re\, L(\bar{v}) \\ &= \inf Re\, L(U) - \sup Re\, L(V) \\ &\leq Re\, L(u) - Re\, L(v) \\ &\leq |L(u-v)| \\ &\leq ||u-v|| .\end{aligned}$$

Hence $\bar{u}, \bar{v}$ are proximal points of U, V respectively.

Theorem 4.4 generalizes Theorem 4.2, which is an extension of a corresponding result of Deutsch and Maserick [4] (also reproved by Havinson [9]) for best approximation from the elements of a convex set. Geometrically, it states that the points $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if there exist a pair of parallel hyperplanes, each of which separates the two convex sets and such that one of them supports U at $\bar{u}$ and the other supports V at $\bar{v}$.

In what follows let us denote by $\mathscr{M}_{(\bar{u},\bar{v})}$, the set

$$\{L \in S_{X^*} / L(\bar{u}-\bar{v}) = ||\bar{u}-\bar{v}||\} .$$

We note that $\mathscr{M}_{(\bar{u},\bar{v})}$ is non-empty by the Hahn-Banach theorem and also $\mathscr{M}_{(\bar{u},\bar{v})}$ is a $\sigma(X^*, X)$—closed (and hence compact), convex, extremal subset of S_{X^*} (cf. Singer [18], pp. 59). As a simple consequence of Theorem 4.4, we obtain the following corollary, which gives a Kolmogorov-type characterization of the proximal points.

Corollary 4.5. Let U, V be convex subsets of X such that $d(U, V)>0$. Then $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if

$$\begin{aligned}&D_{(u,\bar{u})} = \{L \in \mathscr{M}_{(\bar{u},\bar{v})} / Re\, L\,(\bar{u}-u) \leq 0\} \neq \Phi \text{ for each } u \in U \text{ and} \\ &\min_{L \in D_{(u,\bar{u})}} \{Re\, L(\bar{u}-u).\ Re\, L(\bar{v}-v)\} \leq 0, \text{ for each } u \in U \text{ and } v \in V.\end{aligned} \tag{4.5}$$

Proof. The necessity follows at once from Theorem 4.4. To prove the sufficiency we note first that $D_{(u,\bar{u})}$ is a $\sigma(X^*, X)$—compact set and the map $L \rightarrow Re\, L(\bar{u}-u).\ Re\, L(\bar{v}-v)$ is $\sigma(X^*, X)$—continuous. Hence by (4.5), for each $u \in U$ and $v \in V$, there exists an $L = L_{(u,v)} \in \mathscr{M}_{(\bar{u},\bar{v})}$ satisfying: $Re\, L(\bar{u}-u) \leq 0$ and $Re\, L(\bar{v}-v) \geq 0$. The remaining proof follows exactly as in the sufficiency part of Theorem 3.4.

In the case when X is a Hibert space, Theorem 4.4 assumes a more convenient form. This is due to the isometric isomorphism between X and its dual X^*. The functional L of Theorem 4.4 corresponds in this case to the vector $(\bar{u}-\bar{v})/||\bar{u}-\bar{v}||$ and this leads to the corollary.

Corollary 4.6. Let U, V be convex subsets of a Hilbert space X and $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if

$$(4.6)\qquad \begin{aligned} &(1)\quad Re\langle v-\bar{v},\ \bar{u}-\bar{v}\rangle\leq 0 \qquad (v\in V)\,,\\ &(2)\quad Re\langle u-\bar{u},\ \bar{u}-\bar{v}\rangle\geq 0 \qquad (u\in U)\,,\end{aligned}$$

or the reverse inequalities hold in (4.6), that is,

$$(4.7)\qquad \begin{aligned} &(1')\quad Re\langle u-\bar{u},\ \bar{u}-\bar{v}\rangle\leq 0 \qquad (u\in U)\quad\text{and}\\ &(2')\quad Re\langle v-\bar{v},\ \bar{u}-\bar{v}\rangle\geq 0 \qquad (v\in V)\,.\end{aligned}$$

Corollary (4.6) although not stated explicitly in the above form is essentially due to Cheney and Goldstein [2]. Again in [2], Cheney and Goldstein prove the following interesting charactrization of proximal points for the case when X is a Hilbert space and U, V are closed convex subsets of X.

Theorem 4.7. (*Cheney and Goldstein*) Let U, V be closed convex subsets of a Hilbert space X. Then the points $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if they are the fixed points of P_UP_V and P_VP_U respectively, that is, $P_UP_V\bar{u}=\bar{u}$ and $P_VP_U\bar{v}=\bar{v}$, where P_U, P_V are the projections onto U, V respectively.

In the proof of Theorem 4.7, Cheney and Goldstein [2] employ the necessity part of the conditions of Corollary 4.6 and the Lipschitz property satisfied by the projection operator, in order to prove the sufficiency part of Theorem 4.7.

In the following, we generalize the above theorem of Cheney and Goldstein to the case when X is a normed linear space whose dual X^* is strictly convex and U, V are arbitrary convex subsets of X such that $d(U, V)>0$. (This includes for example the spaces $\mathscr{L}_p$, $1<p<\infty$, $p\neq 2$, which are not Hilbert spaces).

Theorem 4.8. Let X be a normed linear space whose dual space X^* is strictly convex and U, V be two convex subsets of X such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal point if and only if $\bar{u}$ is the nearest point (projection point) of $\bar{v}$ in U and $\bar{v}$ is the nearest point (projection point) of $\bar{u}$ in V, that is,

$$(4.8)\qquad \begin{aligned} &||\bar{u}-\bar{v}||=\inf_{u\in U}||u-\bar{v}|| \quad\text{and}\\ &||\bar{u}-\bar{v}||=\inf_{v\in V}||\bar{u}-v||\,.\end{aligned}$$

Proof. The necessity is obvious. To prove the sufficiency, we need the following lemma:

Lemma 4.9. Let X^* be strictly convex and $0 \neq x \in X$. There exists at most one $L \in X^*$, $||L||=1$ such that $L(x)=||x||$.

We censider the hyperplane

$\mathscr{H}=\{L \in X^*/\hat{x}(L)=||x||\}$ in X^*. If there exists an $L \in X^*$ such that $||L||=1$ and $\hat{x}(L)=||x||$, then $d(0, \mathscr{H})=1$. For if $L_1 \in \mathscr{H}$, then $||x|| \; ||L_1|| \geq \hat{x}(L_1)=||x||$. Hence $||L_1|| \geq 1$. Since X^* is strictly convex, there exists at most one element of $\mathscr{H}$ where $d(0, \mathscr{H})$ is attained and the lemma is proved.

Now suppose that $\bar{u} \in U$, $\bar{v} \in V$ satisfy (4.8) then $\bar{u}, \bar{v}$ are proximal points of the sets U, $\{\bar{v}\}$ as well as of the sets $\{\bar{u}\}$, V. Hence by Theorem 4.4 there exist $L_1, L_2 \in X^*$ satisfying

$$(4.1) \quad \begin{array}{ll} (1) & L_1, L_2 \in S_{X^*}, \\ (2) & Re\, L_1(\bar{u}-u) \leq 0 \quad (u \in U), \quad \text{and} \\ & Re\, L_2(\bar{v}-v) \geq 0 \quad (v \in V), \\ (3) & L_1(\bar{u}-\bar{v})=L_2(\bar{u}-\bar{v})=||\bar{u}-\bar{v}||, \end{array}$$

(1) and (3) of (4.9) and Lemma 4.9 gives $L_1=L_2=L$ say. Hence (4.4) is satisfied and by Theorem (4.4) $\bar{u}, \bar{v}$ are proximal points of U, V. This completes the proof.

We note that the necessity part in Theorem 4.8 is obviously true for an arbitrary normed linear space X, however, the sufficiency part may fail to hold if the dual space X^* is not strictly convex.

Examples. (1) Let $X=\mathscr{C}[0, 1]$ with the supremum. Take

$$U=\{(1-\alpha)+\alpha t/0 \leq \alpha \leq 1\} \quad \text{and} \quad V=\{\beta t/0 \leq \beta \leq 1\}.$$

Here $d(U, V)=0$ and the proximal points correspond to $\alpha=\beta=1$.

$$\begin{aligned} ||(1-\alpha)+\alpha t-\beta t|| &= 1-\alpha, \quad \text{if } \alpha \leq \beta, \\ &= 1-\beta, \quad \text{if } \alpha > \beta. \end{aligned}$$

For any $r, 0 \leq r \leq 1$, the points $(1-r)+rt$ of U and rt of V are nearest points to each other from the other set but the points corresponding to $r=1$ are the only proximal points.

(2) Let $X=\boldsymbol{R}^2$ with the norm $||(x_1, x_2)||=\max(|x_1|, |x_2|)$. Take

$$U=\{(\alpha, 0)/1 \leq \alpha \leq 2\} \quad \text{and} \quad V=\{(0, \beta)/1 \leq \beta \leq 2\}.$$

Heren $d(U, V)=1$ and the proximal points correspond to $\alpha=\beta=1$. For a fixed $\alpha_0, 1 \leq \alpha_0 \leq 2$, all the points $(0, \beta)$, $\beta \leq \alpha_0$ of V are the nearest points to the point $(\alpha_0, 0)$ of U and likewise for a fixed $\beta_0, 1 \leq \beta_0 \leq 2$, all the points $(\alpha, 0)$, $\alpha \leq \beta_0$ of U are the nearest points to the point $(0, \beta_0)$ of V. Thus for any $r, 1 \leq r \leq 2$, the points $(r, 0)$, $(0, r)$ of U and V are the nearest points to each other from the other set. However, the points corresponding to $r=1$ are the only proximal points. In the same example, if we take U, V as the open line segments corresponding to $0<\alpha<1$ and $0<\beta<1$ respectively. Then $d(U, V)=1$, the proximal points do not exist and as before for any $r, 1\leq r\leq 2$, the points $(r, 0)$, $(0, r)$ are the projection points of each other.

The particular case, when one of the two convex sets U, V is either a convex cone or a linear subspace, is of considerable interest. In this case, one requires the following easily established facts:

If V is a cone (i. e. $\alpha v\in V$ whenever $v\in V$ and $\alpha\geq 0$) and $\sup Re\, L(V)<\infty$, then $\sup Re\, L(V)=0$. Hence, in particular, if in Theorem 4.4 V is a convex cone, then the separating hyperplane given by $L\in X^*$ supports V at the origin. Also if V is a linear subspace and either $\inf Re\, L(V)>-\infty$ or $\sup Re\, L(V)<\infty$, then $L\in V^{\perp}\equiv\{L\in X^*/L(v)=0 \text{ for all } v\in V\}$. Moreover, if U, V are both cones (or linear subspaces), then $d(U, V)=0$. Thus in the case, when V is a subspace (resp. convex cone) and U is a convex set such that $d(U, V)>0$, Theorem 4.4 reduces to the following:

Theorem 4.10. Let X be an arbitrary normed linear space, U be a convex set and V be a subspace (resp. convex cone) such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if there exists an $L\in X^*$ such that

$$\text{(4.10)}\qquad \begin{aligned} &\text{(i)}\quad L\in S_{X^*}\,, \\ &\text{(ii)}\quad Re\, L(\bar{u}-u)\leq 0 \qquad (u\in U) \quad \text{and} \\ &\qquad\;\; L\in V^{\perp} \quad (\text{resp. } Re\, L(\bar{v})=0)\,, \\ &\text{(iii)}\quad L(\bar{u})=||\bar{u}-\bar{v}||\,. \end{aligned}$$

Theorem 4.10 combines (and generalizes) the well known characterizations of best approximations from the elements of a linear subspace (Singer [13]) and best approximations from the elements of a convex set (Deutsch and Maserick [4] and Havinson [9]).

As applications of Theorem 4.10, we consider characterizations of proximal points in the case of the space $L_p(E, \Sigma, \mu)$, $1\leq p<\infty$. Here let (E, Σ, μ) denote a σ-finite measure space and let $L_p(E, \Sigma, \mu)$. $1\leq p<\infty$ (resp. $p=\infty$) denote the space of complex valued functions p-th power μ-integrable (resp. μ measurable and μ-essentially bounded on E), endowed with the norm:

$$||x||=\left(\int_E |x(t)|^p\, d\mu\right)^{1/p} \quad (\text{resp. } ||x||=\operatorname*{ess.\,sup}_{t\in E} |x(t)|)\,.$$

In what follows, let $Z(x)$ denote the set $\{t\in E/x(t)=0\}$. Firstly, we consider $X=L_1(E, \Sigma, \mu)$. In this case we have.

Lemma 4.11. Let U be a convex set and V be the subspace spanned by a fixed element $v\in X$, such that $d(U, V)>0$. Then $\bar{u}\in U$, $0\in V$ are proximal points if and only if

(i) $\left|\int_{E\setminus Z(\bar{u})} v \operatorname{sgn} \bar{u}\, d\mu\right| \leq \int_{Z(\bar{u})} |v| d\mu$,

(ii) For each $u\in U$ we have

$$Re\int_{E\setminus Z(\bar{u})}(\bar{u}-u)\operatorname{sgn}\bar{u}\,d\mu\leq 0 \text{ if } \int_{Z(\bar{u})}|v|d\mu=0 \text{ and} \tag{4.11}$$

$$\int_{Z(\bar{u})}|v|d\mu\cdot Re\int_{E\setminus Z(\bar{u})}(\bar{u}-u)\operatorname{sgn}\bar{u}\,d\mu\leq Re\left\{\int_{E\setminus Z(\bar{u})}v\operatorname{sgn}\bar{u}\,d\mu \times\int_{Z(\bar{u})}(\bar{u}-u)\operatorname{sgn} v\,d\mu\right\},$$

if $\int_{Z(\bar{u})}|v|d\mu>0$.

(Here $\operatorname{sgn}\alpha=\bar{\alpha}/|\alpha|$ if $\alpha\neq 0$, $\operatorname{sgn}\alpha=0$ if $\alpha=0$).

The lemma follows readily from Theorem 4.10 and the fact that $L_1^*=L_\infty$. In fact, following arguments similar to those in Singer [16], we easily note that the function $h\in L_\infty$ which corresponds to the functional L of Theorem 4.10 is given by:

$$h(t)=\operatorname{sgn}\bar{u}(t) \text{ if } \int_{Z(\bar{u})}|v|d\mu=0 \text{ and if } \int_{Z(\bar{u})}|v|d\mu>0 \text{ then}$$

$$h(t)=\begin{cases}\operatorname{sgn}\bar{u}(t) & \text{if } t\in E\setminus Z(\bar{u}),\\ -\dfrac{\int_{E\setminus Z(\bar{u})}v\operatorname{sgn}(\bar{u})d\mu}{\int_{Z(\bar{u})}|v|d\mu}\operatorname{sgn} v(t) & \text{if } t\in Z(\bar{u}).\end{cases}$$

Lemma 4.11 generalizes the 'basic variational lemma' of Kripke and Rivlin [11] which was used by them as the basic tool in their detailed exposition on L_1 approximation. Lemma 4.11 leads to the following characterization theorem for proximal points in L_1.

Theorem 4.12. Let U be a convex set and V be a subspace in L_1 such that $d(U, V)>0$. Then the following statements are equivalent.

(1) $\bar{u} \in U$, $\bar{v} \in V$ are proximal points.

(2) we have

$$\text{(i)}\quad \left|\int_{E\setminus Z(\bar{u}-\bar{v})} v \operatorname{sgn}(\bar{u}-\bar{v})\, d\mu\right| \leq \int_{Z(\bar{u}-\bar{v})} |v| d\mu \quad \text{for each } v \in V,$$

$$(4.12)\qquad \text{(ii)}\quad Re\int_{E\setminus Z(\bar{u}-\bar{v})} (\bar{u}-u) \operatorname{sgn}(\bar{u}-\bar{v}) d\mu \leq \frac{1}{\int_{Z(\bar{u}-\bar{v})} |v| d\mu}$$

$$\times Re\left\{\int_{E\setminus Z(\bar{u}-\bar{v})} v \operatorname{sgn}(\bar{u}-\bar{v})\, d\mu \int_{Z(\bar{u}-\bar{v})} (\bar{u}-u) \operatorname{sgn}(\bar{u}-\bar{v}) d\mu\right\}.$$

for each $u \in U$ and $v \in V$.
(Here we adopt the convention $0/0=0$ in the right hand side of (ii)).

(3) We have (4.12) (i) and (ii) with $Z(\bar{u}-\bar{v})$ replace by $P(\bar{u}-\bar{v})$ where

$$P(\bar{u}-\bar{v}) = Z(\bar{u}-\bar{v}) \setminus \bigcap_{v \in V} Z(v).$$

The implication (1)$\Longleftrightarrow$(2) follows at once from Lemma 4.11 and the fact that $\bar{u} \in U$, $\bar{u} \in V$ are proximal points if and only if $\bar{u}-\bar{v} \in U-V$ (convex) and $0 \in [\{v\}]$ are proximal points for each $v \in V$ (here and in other places $[\{v\}]$ denotes the linear span of $\{v\}$). The implication (2)$\Longleftrightarrow$(3) follows easily from an argument similar to that of Singer [18] (see Theorem 1.7, pp. 46).

We now consider the spaces $L_p(E, \Sigma, \mu)$, $1<p<\infty$ and in the following establish results analogous to Lemma 4.11 and Theorem 4.12 for these spaces.

Lemma 4.13. Let U be a convex set and V be the subspace spanned by a fixed element v in L_p, $1<p<\infty$, such that $d(U, V)>0$. Then $\bar{u} \in U$, $0 \in V$ are proximal points if and only if

$$(4.14)\qquad \begin{aligned} &\text{(i)}\quad Re\int_E (\bar{u}-u)|\bar{u}|^{p-1} \operatorname{sgn} \bar{u}\, d\mu \leq 0 \qquad (u \in U), \\ &\text{(ii)}\quad \int_E v|\bar{u}|^{p-1} \operatorname{sgn} \bar{u}\, d\mu = 0. \end{aligned}$$

From the fact that $L_p^* = L_q$ where $(1/p)+(1/q)=1$ and the conditions under which equality holds in Hölder's inequality, one readily notes that the functional L of Theorem 4.10 corresponds to the element

$$z = \frac{|\bar{u}|^{p-1} \operatorname{sgn} \bar{u}}{||\bar{u}||^{p/q}} \text{ of } L_q.$$

The lemma now easily follows from Theorem 4.10.

Theorem 4.14. Let U be a convex set and V be a subspace in L_p, $1<p<\infty$, such that $d(U, V)>0$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if

(i) $$Re\int_E(\bar{u}-u)|\bar{u}-\bar{v}|^{p-1}\operatorname{sgn}(\bar{u}-\bar{v})d\mu\leq 0 \qquad (u\in U),$$

(ii) $$\int_E v|\bar{u}-\bar{v}|^{p-1}\operatorname{sgn}(\bar{u}-\bar{v})d\mu=0 \qquad (v\in V).$$

The theorem follows immediately from the Lemma 4.13 since $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if $\bar{u}-\bar{v}\in U-V$, $0\in[\{v\}]$ are proximal points for each $v\in V$.

5. Characterization of Proximal Points Using Extreme Functionals

Here we first note that in the (important) special case when the convex sets U, V are contained in a finite-dimensional subspace of X, Theorems 3.2 and 4.4 (and hence the majority of results of the last section) can be substantially improved. For this we need the following:

Lemma 5.1. Let X be a Banach space of finite dimension n and let $L\in K_{X^*}$. Then there exist m functionals $L_i\in\mathscr{E}(K_{X^*})$, $i=1, 2, \cdots, m$ and m numbers $\lambda_1,\cdots,\lambda_m>0$ with $\sum_{i=1}^m\lambda_i=1$, such that $L=\sum_{i=1}^m\lambda_iL_i$. Here $1\leq m\leq n$ if the scalars are real and $1\leq m\leq 2n-1$ if the scalars are complex.

The lemma is well known (see for example Singer [18], pp. 166). It follows from an inductive argument similar to that in Caratheodory theorem.

Lemma 5.2. Let G be a subspace of a normed linear space X and let Φ be an extreme point of the closed unit ball K_{G^*} of G^*. Then Φ has an extension $L\in\mathscr{E}(K_{X^*})\subseteq X^*$.

Lemma 4.2 is an extension theorem due to Singer [15]. It follows from the Krein-Milman theorem and the fact that the set of all norm preserving extensions of Φ, $\{L\in X^*/||L||=1 \text{ and } L|_G=\Phi\}$ is a non-void, $\sigma(X^*, X)$—compact convex extremal subset of the unit ball K_{X^*}.

Lemma 5.3. Let the convex sets U, V be such that $[U\cup V]$ is an n-dimensional subspace of X. Then the functional L of Theorem 3.2 (hence also of Theorem 4.4 and all other results of the last section which depend on this) can be expressed as

(5.1) $$L=\sum_{j=1}^m\lambda_iL_i,$$

where

$L_i \in \mathscr{E}(K_{X^*})$, $i=1, 2, \cdots, m$, $\lambda_i>0$, $i=1, \cdots, m$ with $\sum_{i=1}^{m} \lambda_i=1$ and where $m\leq n$ if the scalars are real, $m\leq 2n-1$ if they are complex.

This follows at once from Lemmas 5.1 and 5.2.

Using Lemma 5.3, Theorem 4.4 reduces in this case to the following:

Theorem 5.4. Let U, V be convex subsets of X such that $d(U, V)>0$ and $\dim[U\cup V]=n$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if there exist m functionals $L_i \in \mathscr{E}(K_X)$, $i=1, \cdots, m$ and m numbers λ_i, $i=1, 2, \cdots, m$ such that

$$\text{(5.2)}\quad \begin{aligned} &\text{(i)}\quad \lambda_i>0\,, \qquad i=1, \cdots, m, \quad \sum_{1}^{m} \lambda_i=1,\\ &\text{(ii)}\quad Re\sum_{i=1}^{m} \lambda_i L_i(\bar{u}-u)\leq 0 \qquad (u\in U)\quad \text{and}\\ &\qquad\quad Re\sum_{i=1}^{m} \lambda_i L_i(\bar{v}-v)\geq 0 \qquad (v\in V)\,,\\ &\text{(iii)}\quad L_i(\bar{u}-\bar{v})=||\bar{u}-\bar{v}||, \qquad i=1, \cdots, m, \end{aligned}$$

where $m\leq n$ if the scalars are real, $m\leq 2n-1$ if they are complex.

Remarks. If in Theorem 5.4, we take V as a subspace then the condition (5.2) (ii) should be replaced by

$$Re\sum_{i=1}^{m} \lambda_i L_i(\bar{u}-u)\leq 0 \qquad (u\in U)\quad \text{and}$$

$$\sum_{i=1}^{m} \lambda_i L_i(v)=0 \qquad (v\in V)\,.$$

This improves a corresponding result of Singer [13]. In the case when $X=\mathscr{C}(T)$, the space of continuous functions (real or complex) defined on a compact Hausdorff space T with the uniform norm: $||x||=\max_{t\in T}|x(t)|$, $x\in X$, Theorem 5.4 assumes a more convenient from. This is due to (the well known) fact that $\mathscr{E}(K_{X^*})=\{\delta_t\Phi_t/t\in T, |\delta_t|=1\}$, where each Φ_t is a point evaluation functional corresponding to t, that is,

$$\Phi_t(x)=x(t)\,, \qquad x\in X\,.$$

We thus have:

Corollary 5.5. Let U, V be convex subsets of $\mathscr{C}(T)$ such that $d(U, V)>0$ and $\dim[U\cup V]=n$. Then $\bar{u}\in U$, $\bar{v}\in V$ are proximal points iff there exist m points $t_i\in T_{(\bar{u},\bar{v})}=\{t\in T/|\bar{u}(t)-\bar{v}(t)|=||\bar{u}-\bar{v}||\}$ and m numbers λ_i such that

$$\text{(i)}\quad \lambda_i>0\,,\qquad \sum_{i=1}^{m}\lambda_i=1\,,$$

(5.3)
$$\text{(ii)}\quad Re\sum_{i=1}^{m}\lambda_i\,\mathrm{sgn}\,[\bar{u}(t_i)-\bar{v}(t_i)][\bar{u}(t_i)-u(t_i)]\leq 0 \qquad (u\in U)\quad\text{and}$$

$$Re\sum_{i=1}^{m}\lambda_i\,\mathrm{sgn}\,[\bar{u}(t_i)-\bar{v}(t_i)][\bar{v}(t_i)-v(t_i)]\geq 0 \qquad (v\in V)\,.$$

(If V is a subspace, then the second part of this condition is to be replaced by:

$$\sum_{i=1}^{m}\lambda_i\,\mathrm{sgn}\,[\bar{u}(t_1)-\bar{v}(t_i)]v(t_i)=0 \qquad (v\in V)\,,$$

where $m\leq n$ if the scalars are real and $m\leq 2n-1$ if they are complex. In the case, when $U=\{\bar{u}\}$ and V is a k-dimensional subspace of $\mathscr{C}(T)$ we get from the above corollary the following:

$\bar{v}\in V$ is a projection point (best approximation) to $\bar{u}$ if and only if the origin $(0,\cdots,0)$ of the k-space C^k is in the convex hull of the set of k-tuples

$$\{\overline{r(t)},\,(v_1(t),\,\cdots,\,v_k(t))/t\in T,\,|\bar{u}(t)-\bar{v}(t)|=||\bar{u}-\bar{v}||\}\,,$$

where $\{v_1,\cdots,v_k\}$ is any basis of V and $r(t)=\bar{u}(t)-\bar{v}(t)$. This is a well known corollary of Cheney and Goldstein [3] and it now turns out to be a special case of Corollary 5.5.

In what follows, we now propose to generalize the Garkavi-type Theorem 4.3. For this we need the following:

Definition. Let U, V be arbitrary convex subsets of X and $(\bar{u},\bar{v})$ where $\bar{u}\in U$, $\bar{v}\in V$ be a fixed pair of points. We call the sets U, V $(\bar{u},\bar{v})$—symmetric if

(5.4)
$$\begin{gathered}\{L\in\mathscr{M}_{(\bar{u},\bar{v})}/Re\,(\bar{u}-u)^\wedge L=\inf Re\,(\bar{u}-u)^\wedge\mathscr{M}_{(\bar{u},\bar{v})}\}\\ \bigcap\\ \{L\in\mathscr{M}_{(\bar{u},\bar{v})}/Re\,(\bar{v}-v)^\wedge L=\sup Re\,(\bar{v}-v)^\wedge\mathscr{M}_{(\bar{u},\bar{v})}\}\\ \neq\Phi,\ \text{for each } u\in U \text{ and } v\in V.\end{gathered}$$

If the set U consists of a single point $\bar{u}$ and $V\subseteq X$ is an arbitrary convex set, then trivially U, V are $(\bar{u},\bar{v})$—symmetric. The following are some nontrivial examples of $(\bar{u},\bar{v})$—symmetric convex sets.

Example 1. Let $X=\boldsymbol{R}^2$ with the norm $||(x_1,x_2)||=|x_1|+|x_2|$,

$$U=\{(x_1,0)/0\leq x_1\leq 1\}\quad\text{and}\quad V=\{(x_1,x_2)/1\leq x_1\leq 2,\,1\leq x_2\leq 2\}\,.$$

Take $\bar{u}=(1,0)$ and $\bar{v}=(1,1)$. $X^*=\boldsymbol{R}^2$ with the norm

$$||(x_1,x_2)||=\max(|x_1|,|x_2|)\,.\quad \mathscr{M}_{(\bar{u},\bar{v})}=\{(\alpha,-1)/|\alpha|\leq 1\}\,.$$

It is readily verified that the functional $L \in \mathscr{M}_{(\bar{u},\bar{v})}$ given by $L=(-1, -1)$ belongs to the set given by (5.4) for each $u \in U$ and $v \in V$.

Example 2. Again take $X=\boldsymbol{R}^2$ with the norm $\|(x_1, x_2)\|=\max(|x_1|, |x_2|)$. Let the sets U, V and the points $\bar{u}, \bar{v}$ be as in Example 1. Then $X^*=\boldsymbol{R}^2$ with the norm $\|(x_1, x_2)\|=|x_1|+|x_2|$ and in this case $\mathscr{M}_{(\bar{u},\bar{v})}=\{(0, -1)\}$. Hence, trivially, the set in (5.4) contains $(0, -1)$ for each $u \in U$ and $v \in V$.

Example 3. Let $X=\mathscr{C}[-1, +1]$ (real functions),

$$U=\{2\alpha|t|+(1-\alpha)t^2/0\leq\alpha\leq1\} \quad \text{and} \quad V=\{-\beta t^4/0\leq\beta\leq1\} .$$

Take $\bar{u}=t^2$, $\bar{v}=0$. It is easily verified that the set in (5.4) contains the point evaluation functionals Φ_{-1} and Φ_{+1} (see also the remark following Theorem 5.11), for each $u \in U$ and $v \in V$.

To prove the next theorem, we shall employ Krein-Milman type reasoning. For this we require the following lemmas which are easy consequences of the definitions of extreme points and extremal sets.

Lemma 5.7. Let Y be an extremal subset of Z and Z be an extremal subset of W. Then Y is an extremal subset of W.

Lemma 5.8. Let M be an extremal subset of a set $A\subseteq X$. Then

$$\mathscr{E}(M)=M\cap \mathscr{E}(A) .$$

Lemma 5.9. Let $L \in X^*$ and A be a subset of X, such that the sets

$$Y=\{x \in A/Re\, L(x)=\inf Re\, L(A)\} ,$$

$$Z=\{x \in A/Re\, L(x)=\sup Re\, L(A)\} \text{ are non-empty.}$$

Then the sets Y and Z are extremal subsets of A.

Theorem 5.10. (Characterization theorem of Garkavi-type) Let U, V be convex subsets of X such that $d(U, V)>0$. Suppose $\bar{u} \in U$, $\bar{v} \in V$ are points such that the sets U, V are $(\bar{u}, \bar{v})$—symmetric. Then $\bar{u}, \bar{v}$ are proximal points of U, V if and only if for each $u \in U$ and $v \in V$, there exists an $L=L_{(u,v)} \in X^*$ such that:

$$\text{(5.5)} \qquad \begin{array}{ll} \text{(i)} & L \in \mathscr{E}(S_{X^*}) , \\ \text{(ii)} & Re\, L(\bar{u}-u)\leq0 \quad \text{and} \\ & Re\, L(\bar{v}-v)\geq0 , \\ \text{(iii)} & L(\bar{u}-\bar{v})=\|\bar{u}-\bar{v}\| . \end{array}$$

Proof. The sufficiency part is proved exactly as in Theorem 4.4. (In fact note that for this part the assumption of the $(\bar{u}, \bar{v})$—symmetry of the sets U, V

is not essential). For the necessity, let $\bar{u} \in U$, $\bar{v} \in V$ be proximal points and assume that the sets U, V are $(\bar{u}, \bar{v})$—symmetric. Now let $u \in U$, $v \in V$ and define the set

$$\mathscr{A}_{(u,v)} = \{L \in \mathscr{M}_{(\bar{u},\bar{v})} / Re\,(\bar{u}-u)^\wedge L = \inf Re\,(\bar{u}-u)^\wedge \mathscr{M}_{(\bar{u},\bar{v})}\} \\ \cap \\ \{L \in \mathscr{M}_{(\bar{u},\bar{v})} / Re\,(\bar{v}-v)^\wedge L = \sup Re\,(\bar{v}-v)^\wedge \mathscr{M}_{(\bar{u},\bar{v})}\}\,.$$

$\mathscr{A}_{(u,v)}$ is non-void by the assumption of $(\bar{u}, \bar{v})$—symmetry of U, V. Also $\mathscr{A}_{(u,v)}$ is a $\sigma(X^*, X)$—closed (and hence compact), convex, extremal subset of $\mathscr{M}_{(\bar{u},\bar{v})}$. Therefore, by Krein-Milman theorem $\mathscr{A}_{(u,v)}$ has an extreme point L_e. Since $\mathscr{A}_{(u,v)}$ is an extremal subset of $\mathscr{M}_{(\bar{u},\bar{v})}$ and $\mathscr{M}_{(\bar{u},\bar{v})}$ is an extremal subset of S_{X^*}, $\mathscr{A}_{(u,v)}$ is an extremal subset of S_{X^*}. Hence $\mathscr{E}(\mathscr{A}_{(u,v)}) = \mathscr{A}_{(u,v)} \cap \mathscr{E}(S_{X^*})$ and $L_e \in \mathscr{E}(S_{X^*})$. By Theorem 4.4, there exists an $L \in \mathscr{M}_{(\bar{u},\bar{v})}$ such that $Re\,L(\bar{u}-u) \leq 0$ $(u \in U)$ and $Re\,L(\bar{v}-v) \geq 0$ $(v \in V)$. Hence $Re\,L_e(\bar{u}-u) \leq Re\,L(\bar{u}-u) \leq 0$ and $Re\,L_e(\bar{v}-v) \geq Re\,L(\bar{v}-v) \geq 0$. L_e thus satisfies (i), (ii), (iii) and the proof is complete.

Remark. We note that for each $u \in U$ and $v \in V$, there exists an $L \in X^*$ satisfying (5.5) if and only if for each $u \in U$ and $v \in V$, there exists an $L \in X^*$ satisfying

(5.6)
$$\begin{aligned} &(\mathrm{i}) \quad L \in \mathscr{E}(S_{X^*})\,, \\ &(\mathrm{ii}) \quad Re\,[\overline{L(\bar{u}-\bar{v})} \cdot L(\bar{u}-u)] \leq 0 \quad \text{and} \\ &\qquad\; Re\,[\overline{L(\bar{u}-\bar{v})} \cdot L(\bar{v}-v)] \geq 0\,, \\ &(\mathrm{iii}) \quad |L(\bar{u}-\bar{v})| = \|\bar{u}-\bar{v}\|\,. \end{aligned}$$

Furthermore, if V is a subspace, this is equivalent to: for each $u \in U$ and $v \in V$, there exists an $L \in X^*$ satisfying (5.6) (i), (iii) and

(5.7)
$$\begin{aligned} &Re\,[\overline{L(\bar{u}-\bar{v})} \cdot L(\bar{u}-u)] \leq 0\,, \\ &Re\,[\overline{L(\bar{u}-\bar{v})} \cdot L(v)] \geq 0\,. \end{aligned}$$

The implication (5.5)$\Longrightarrow$(5.6) is obvious. (5.6)$\Longrightarrow$(5.5) follows by putting $f = \operatorname{sgn} L(\bar{u}-\bar{v}) \cdot L$. (5.6)$\Longrightarrow$((5.7) follows by taking $\Phi_{(u,v)} = L_{(u,\bar{v}-v)}$. The implication (5.7)$\Longrightarrow$(5.6) is similar (the equivalences are established by arguments analogous to those in Singer [18], pp. 63 Corollary 1.9).

If in Theorem 5.10, one takes $U = \{\bar{u}\}$ and V as an arbitrary convex subset of X, then one recovers as a special case the Garkavi theorem for best approximation from the elements of a convex set. In [8], Garkavi in fact proved this result for a Banach space and again in [4] Deutsch and Maserick reproved the same for an arbitrary normed linear space.

We next consider the case when $X=\mathscr{C}(T)$ (T compact Hausdorff) and obtain as an easy consequences of Theorem 5.10, the following Kolmogorov-type characterization of proximal points.

Theorem 5.11. (*Kolmogorov-type characterization*) Let U, V be convex subsets of $\mathscr{C}(T)$ such that $d(U, V)>0$. Suppose $\bar{u}\in U$, $\bar{v}\in V$ are points such that U, V are $(\bar{u}, \bar{v})$—symmetric. Then $\bar{u}, \bar{v}$ are proximal points of U, V if and only if for each $u\in U$, $v\in V$, there exists a $t=t_{(u,v)}\in T$ such that

$$(5.8)\qquad \begin{aligned} &(\mathrm{i}) && Re\{[\overline{\bar{u}(t)-\bar{v}(t)}][\bar{u}(t)-u(t)]\}\leq 0\ ,\\ &(\mathrm{ii}) && Re\{[\overline{\bar{u}(t)-\bar{v}(t)}][\bar{v}(t)-v(t)]\}\geq 0 \quad\text{and}\\ &(\mathrm{iii}) && |\bar{u}(t)-\bar{v}(t)|=||\bar{u}-\bar{v}||=\max_{t\in T}|\bar{u}(t)-\bar{v}(t)|\ .\end{aligned}$$

(As in the remark preceeding this theorem, it V is a subspace (ii) can be replaced by $Re\{[\overline{\bar{u}(t)-\bar{v}(t)}]v(t)\}\geq 0$).

Proof. By Theorem 5.10 and the general form of the extreme points of S_{X^*}, we have $\bar{u}\in U$, $\bar{v}\in V$ are proximal points if and only if for each $u\in U$, $v\in V$, there exists a $t=t_{(u,v)}\in T$ and a scalar $\delta=\delta_{(u,v)}$ with $|\delta|=1$, such that

$$(5.9)\qquad \begin{aligned} &Re\{\delta[\bar{u}(t)-u(t)]\}\leq 0\ ,\\ &Re\{\delta[\bar{v}(t)-v(t)]\}\geq 0 \quad\text{and}\\ &\delta[\bar{u}(t)-\bar{v}(t)]=||\bar{u}-\bar{v}||\ .\end{aligned}$$

The last equation gives $\delta=\operatorname{sgn}[\bar{u}(t)-\bar{v}(t)]$ and hence (5.9) is equivalent to (5.8). This completes the proof.

Remark. We note that in this case the convex sets U, V are $(\bar{u}, \bar{v})$—symmetric iff they satisfy

$$(5.10)\qquad \begin{aligned} &\{\bar{t}\in T/Re([(\bar{u}-\bar{v})(\bar{t})][(\bar{u}-u)(\bar{t})])=\min_{t\in T_{(\bar{u},\bar{v})}} Re([(\bar{u}-\bar{v})(\bar{t})][(\bar{u}-u)(t)])\}\\ &\qquad\qquad\qquad\qquad\bigcap\\ &\{\bar{t}\in T/Re([(\bar{u}-\bar{v})(\bar{t})][(\bar{v}-v)(\bar{t})])=\max_{t\in T_{(\bar{u},\bar{v})}} Re([\bar{u}-\bar{v})(\bar{t})][(\bar{v}-v)(t)])\}\ ,\end{aligned}$$

$\neq\Phi$, for each $u\in U$ and $v\in V$. Here $T_{(\bar{u},\bar{v})}=\{t\in T/|\bar{u}(t)-\bar{v}(t)|=||\bar{u}-\bar{v}||\}$. This follows at once from the general form of the extreme points of S_{X^*} and the well known fact that a real continuous linear functional L attains its infimum (resp. supremum) on a compact convex set K at an extreme point of K. We need only observe here that $(\bar{u}-u)^\wedge$, $(\bar{v}-v)^\wedge$ are $\sigma(X^*, X)$—continuous and $\mathscr{M}_{(\bar{u},\bar{v})}$ is a $\sigma(X^*, X)$—compact convex set. Again in Theorem 5.11, if one takes $U=\{\bar{u}\}$ and V as an arbitrary linear subspace of X, then (5.10) is trivially satisfied and one obtains as a special case the well known Kolmogorov theorem (see [10]) for the characterization of best approximation from elements of linear subspaces of $\mathscr{C}(T)$.

Now suppose that $\mathscr{L}=\mathscr{M}_{(\bar{u},\bar{v})}\cap\Gamma$, where Γ is a fundamental system (following Nikolskii [12]), that is, a $\sigma(X^*, X)$—closed subset of K_{X^*} such that for each $x\neq 0$ in X there exists an $L\in\Gamma$ with $L(x)=||x||$. Examples of fundamental systems are the unit sphere S_{X^*} and $\mathscr{E}(K_{X^*})$. Assume further that U, V satisfy

$$(5.11)\quad \begin{aligned}&\{L\in\mathscr{L}/Re([\overline{(\bar{u}-\bar{v})^\wedge L}]\cdot[(\bar{u}-u)^\wedge L])=\min_{L\in\mathscr{L}} Re([\overline{(\bar{u}-\bar{v})^\wedge L}][(\bar{u}-u)^\wedge L])\}\\ &\{L\in\mathscr{L}/Re([\overline{(\bar{u}-\bar{v})^\wedge L}]\cdot[(\bar{v}-v)^\wedge L])=\max_{L\in\mathscr{L}} Re([\overline{(\bar{u}-\bar{v})^\wedge L}][(\bar{v}-v)^\wedge L])\}\,,\end{aligned}$$

$\neq\Phi$, for each $u\in U$ and $v\in V$. Then since the cannonical mapping $x\to\hat{x}$ of X into $\mathscr{C}(\Gamma)$, defined by

$$\hat{x}(L)=L(x)\qquad(L\in\Gamma,\ x\in X)$$

is an isometric isomorphism, we readily obtain from Theorem 5.10.

Corollary 5.12. Let $\Gamma\subseteq K_{X^*}$ be a fundamental system and suppose that the convex sets U, V and the points $\bar{u}\in U$, $\bar{v}\in V$ satisfy (5.11). Then $\bar{u}$, $\bar{v}$ are proximal points of U, V if and only if for each $u\in U$, $v\in V$ there exists an $L=L_{(u,v)}\in\Gamma$ satisfying (5.6) (ii) and (iii) (and satisfying (5.7) (ii) ann (iii) if V is assumed to be a subspace).

Corollary 5.11 generalizes a theorem of Nikolskii [12]. In fact, if we take $U=\{\bar{u}\}$ and V as an arbitrary convex set we recover the theorem of Nikolskii [12].

Finally, we shall obtain a result corresponding to Theorem 5.11 for the case when X is a linear subspace of $\mathscr{C}(T)$ (T compact Hausdorff). Recall that (for this case) the Choquet boundary of X is the set

$$\gamma(X)=\{\bar{t}\in T/\mu[A_{\bar{t}}(X)]=1\quad(\mu\in\mathscr{O}_{\bar{t}}(X))\}\,,\quad\text{with}$$

$$A_{\bar{t}}(X)=\{t\in T/x(\bar{t})=x(t)\quad(x\in X)\}\quad\text{and}$$

$$\mathscr{O}_{\bar{t}}(X)=\left\{\mu\Big/\int_T x(t)\,d\mu(t)=x(\bar{t})\quad(x\in X),\ \mu(T)=1\right\}\,,$$

(μ being a positive Radon measure on T). The general form of extreme points of the unit ball K_{X^*} in this case is given by the following:

$$(5.12)\qquad \mathscr{E}(K_{X^*})\subseteq\{\delta_t\Phi_t/t\in\gamma(X)\,,\quad|\delta_t|=1\}\,,$$

where $\gamma(X)$ is the Choquet boundary of X and Φ_t as before is the point evaluation functional corresponding to point t. Furthermore, equality occurs in the above inclusion if $1\in X$ (where 1 denotes the function$\equiv$1 on T). The above representation of extreme points of K_{X^*} is due to Singer (cf. [17], Theorem 1). Using this representation, we now obtain the following:

Theorem 5.13. Let X be a linear subspace of $\mathscr{C}(T)$ (T compact Hausdorff) and $\gamma(X)$ be the Choquet boundary of X. Suppose the convex subsets U, V of X and the points $\bar{u} \in U$, $\bar{v} \in V$ satisfy (5.10), with $T_{(\bar{u},\bar{v})}$ replaced by $T_{(\bar{u},\bar{v})} \cap \gamma(X)$. Then $\bar{u} \in U$, $\bar{v} \in V$ are proximal points if and only if for each $u \in U$, $v \in V$ there exists a $t = t_{(u,v)} \in \gamma(X)$ such that we have (5.8).

Again if in Theorem 5.13, we take $U = \{\bar{u}\}$ and V as a linear subspace of X, then we obtain as a particular case a theorem of Singer (cf. [17], Theorem 3).

6. Uniqueness of Proximal Points

Here we are mainly concerned with conditions under which for a given pair U, V of convex sets, there exists at most one (resp. unique) pair $\bar{u}, \bar{v}$ of proximal points. We shall call a pair U, V of convex sets, a *proximinal pair* if there exists a pair $\bar{u}, \bar{v}$ of proximal points. If there exists at most one pair of proximal points for U, V, then the pair U, V is called a *semi-Chebyshev pair* (following Efimov and Stečkin [6], who actually introduced the term 'Chebyshev subspace'). If the pair U, V is proximinal as well as semi-Chebyshev, then it is called a *Chebyshev pair*. Each non-proximinal pair is obviously semi-Chebyshev. Also if X is strictly convex and U, V satisfy $(U-U) \cap (V-V) = \{0\}$, then as in Theorem 3.1 it is easily established that the pair U, V is semi-Chebyshev. Moreover, if X is a uniformly convex Banach space and U, V are closed convex sets, one of them being compact, then by Theorem 3.1 the pair U, V is Chebyshev. However, one can have a Chebyshev pair of convex sets even for non-uniformly convex spaces. It is well known for example (classical Chebyshev-theory) that if we take $X = \mathscr{C}(T)$, $U = \{\bar{u}\}$ and V as an n-dimensional Haar subspace of $\mathscr{C}(T)$ then the pair U, V is Chebyshev. On the other hand, it is easy to construct examples of convex sets in $\mathscr{C}(T)$ for which there are infinitely many pairs of proximal points, e.g., let $X = \mathscr{C}[0, 1]$ (real functions) $U = \{(1-\alpha) + \alpha t^2 / 0 \leq \alpha \leq 1\}$, $V = \{-\beta t / 0 \leq \beta \leq 1\}$. Then the points $(1-\alpha) + \alpha t^2 \in U$, $0 \in V$ are proximal points ($0 \leq \alpha \leq 1$).

The following theorem gives necessary and sufficient conditions in order that the pair $\bar{u}, \bar{v}$ of points of U, V be the unique pair of proximal points.

Theorem 6.1. Let U, V be a proximinal pair of convex sets and let $\bar{u}, \bar{v}$ be a pair of proximal points. Then U, V is a Chebyshev pair if and only if there do not exist $u \in U \backslash \{\bar{u}\}$, $v \in V \backslash \{\bar{v}\}$ and an $L \in X^*$ such that

(6.1)
(i) $L \in S_{X^*}$,
(ii) $L(u)=L(\bar{u})$ and $L(v)=L(\bar{v})$,
(iii) $L(u-v)=||u-v||$.

Theorem 6.1 follows immediately from Theorem 4.4.

Remark. We readily observe that 6.1 (iii) may be replaced by $|L(u-v)| =||u-v||$. If we take $U=\{\bar{u}\}$ and V as a subspace, we get a result of Singer [13]. In [14], Singer has used this to good effect in characterizing all the Chebyshev subspaces of $L_1[a, b]$. The next theorem which also follows readily from Theorem 4.4 would seem to be useful in characterizing semi.Chebyshev pairs of convex sets.

Theorem 6.2. A pair U, V of convex subsets of X is semi-Chebyshev if and only if there does not exist an $L \in X^*$ such that

(i) $L \in S_{X^*}$,

(ii) $Re\, L(u_i)=\inf Re\, L(U)$ for two distinct $u_i \in U$, $i=1, 2$ and $Re\, L(v_i)= \sup Re\, L(V)$ for two distinct $v_i \in V$, $i=1, 2$,

(iii) $L(u_i-v_i)=||u_i-v_i||$, $i=1,2$.

7. Duality Results for Distance Between Convex Sets

In this section, we propose to study the problem of determining the distance between two convex sets. To this end we first reformulate Theorem 3.2 as a duality theorem which expresses this problem as an equivalent maximization problem in the dual space. Geometrically speaking, it states that the distance between two convex sets can be expressed as the maximum of the distances between pairs of parallel hyperplanes each of which separates the two convex sets.

Theorem 7.1. (Duality Theorem) Let U, V be convex subsets of X such that $d(U, V)>0$. Then

$$d(U, V)=\max_{L \in S_{X^*}} \{\inf Re\, L(U)-\sup Re\, L(V)\}. \tag{7.1}$$

This follows at once from Lemma 3.4 (i) and Theorem 3.2. Theorem 7.1 generalizes a corresponding duality theorem of Garkavi [7] for $d(x, K)$ where K is a convex set and $x \notin d(K)$. It is useful in determining lower bounds on $d(U, V)$.

Remark. It is obvious that one could restrict the maximum in (7.1) over the smaller set $\{L \in S_{X^*} / \sup Re\, L(V) \leq \inf Re\, L(U)\}$.

Colollary 7.2. If $0 \in V$, then

$$d(U, V)=\max_{L\in\mathscr{L}} \{\inf Re\, L(U)-\sup Re\, L(V)\} \quad \text{where} \tag{7.2}$$
$$\mathscr{L}=\{(\inf_{L\in U} ||u||\cdot V^{\circ})\cap S_{X^*}\} .$$

Heren V^0 denotes the polar of V, i.e. $V^{\circ}=\{L\in X^*/\sup Re\, L(V)\leq 1\}$.

Proof. If $\inf_{u\in U} ||u||=0$, the result is obvious. Otherwise, by the remark preceeding this corollary, the maximum may be taken over all $L\in S_{X^*}$ satisfying $\sup Re\, L(V)\leq \inf Re\, L(U)$. For each such L, $\sup Re\, L(V)\leq \inf_{u\in U} ||u||$ i.e., $L\in \inf_{u\in U} ||u||\cdot V^0$ and the proof is complete.

In the above corollary $0\in V$ is no restriction at all, since upon translating V by a vector $v\in V$, the translate $V-v$ is convex, contains 0 and $d(U-v, V-v) =d(U, V)$.

Corollary 7.3. If V is a convex cone (resp. a subspace) and U is a convex set such that $d(U, V)>0$. Then $d(U, V)=\max\{\inf Re\, L(U)\}$, where the maximum is taken over $L\in S_{X^*}$ satisfying $\sup Re\, L(V)=0$ (resp. $L\in V^{\perp}$).

Next we consider the case when U, V are convex subsets of the dual space $E^*=X$. Theorem 7.1 of course applies here also, but it is often more convenient to know if one could restrict the search for the maximum over certain $L\in E^{**}$ to $\sigma(E^*, E)$—continuous linear functionals in E^{**} (for a possibly non-reflexive E). The following theorem gives sufficient conditions for this to be valid.

Theorem 7.4. *Let U^*, V^* be two $\sigma(X^*, X)$—closed convex subsets of X^* such that U^* is norm bounded and $d(U^*, V^*)>0$. Then,*

$$d(U^*, V^*)=\sup[\inf Re\, \hat{x}(U^*)-\sup Re\, \hat{x}(V^*)] , \tag{7.3}$$

The supremum being taken over all $x\in X$, $||x||=1$.

Proof. For each integer $n\geq 2$, let

$$A_n^*=\{L\in X^*/d(L, U^*)\leq(1-n^{-1})d(U^*, V^*)\} .$$

Then A_n^* is convex, $\sigma(X^*, X)$—closed and norm-bounded, hence $\sigma(X^*, X)$—compact. Also $U^*\subseteq A_n^*$ and $A_n^*\cap V^*=\Phi$. Thus (cf. [5], p. 417) there exists a $\sigma(X^*, X)$—continuous linear functional i.e. an $\hat{x}_n\in \hat{X}$, such that $||\hat{x}_n||=1$ and

$$\sup Re\, \hat{x}_n(V^*)<\inf Re\, \hat{x}_n(A_n^*)\leq \inf Re\, \hat{x}_n(U^*) .$$

Let

$$H_n^*=\{L\in X^*/ Re\, \hat{x}_n(L)=\sup Re\hat{x}_n(V^*)\} .$$

Then H_n^* is disjoint from A_n^* and therefore $d(L, U^*)>(1-n^{-1})d(U^*, V^*)$ for each $L \in H_n^*$. Hence $d(H_n^*, U^*) \geq (1-n^{-1})d(U^*, V^*)$. Thus using Lemma 3.3,

$$\sup_n [\inf Re\, \hat{x}_n(U^*) - \sup Re\, \hat{x}_n(V^*)] = \sup_n d(H_n^*, U^*) = d(U^*, V^*) ,$$

and the proof is complete.

Corollary 7.5. Let U^*, V^* be as in Theorem 7.4 and suppose $0 \in V^*$. Then

$$d(U^*, V^*) = \sup \{\inf Re\, \hat{x}(U^*) - \sup Re\, \hat{x}(V^*)\} ,$$

The supremum being taken over all

$$x \in \inf_{L \in U^*} ||L||^0(V^*) \quad \text{with} \quad ||x||=1 .$$

(Here we define ${}^0(V^*) = \{x \in X / \sup Re\, \hat{x}(V^*) \leq 1\}$).

The proof is analogous to the proof of Corollary 7.2.

Corollary 7.6. Let U^* be as in Theorem 7.4 and suppose

$$V^* = \{L \in X^* / \sup Re\, L(V) = 0\} \ (\text{resp. } V^* = V^{\perp})$$

where V is a convex cone (resp. a subspace) in X. Then

$$d(U^*, V^*) = \sup \{\inf_{L \in U^*} L(x)\} , \tag{7.4}$$

The supremum being taken over all $x \in V$ with $||x||=1$.

Proof. The proof follows readily by noting that if V is a convex cone, then $V = {}^0(V^*)$; while if V is a subspace, then $V^* = V^{\perp}$.

In general, the 'sup' in Theorem 7.4 (and hence also in Corollaries 7.5 and 7.6) cannot be always replaced by the maximum. This is evident from the following example. Let $X = l_1$. Then $X^* = l_\infty$ and take $U^* = \{(1, 1, \cdots)\}$, $V^* = \{(1/2, 1/3, \cdots, 1/n+1, \cdots)\}$. Here $d(U^*, V^*) = 1$, but for each $x = (x_1, x_2, \cdots) \in l_1$ with $||x|| = \Sigma |x_n| = 1$, we have

$$\inf Re\, \hat{x}(U^*) - \sup Re\, \hat{x}(V^*) = Re\, \Sigma(n/n+1)x_n < 1 .$$

In the next theorem, we embed the problem of finding the distance between two convex sets into the second dual space and obtain in particular, a sufficient condition under which 'sup' in Theorem 7.4 can be replaced by 'max'.

Theorem 7.7. Let U, V be convex subsets of X such that $d(U, V)>0$. Then,

$$\begin{aligned} d(U, V) &= d(\hat{U}, \tilde{V}) = \max [\inf Re\, \hat{L}(\hat{U}) - \sup Re\, \hat{L}(\tilde{V})] \\ &= d(\tilde{U}, \hat{V}) = \max [\inf Re\, \hat{L}(\hat{V}) - \sup Re\, \hat{L}(\tilde{U})] , \end{aligned} \tag{7.5}$$

The maximum being taken over $L \in S_{X^*}$. Here $\tilde{U}$, $\tilde{V}$ denote the $\sigma(X^{**}, X^*)$—closures (or X^{**}—closures) of $\hat{U}$, $\hat{V}$ in X^{**}.

Proof. Since the natural embedding is an isometry, $d(U, V)=d(\hat{U}, \hat{V})$ and by Theorem 3.2, there exists an $L \in X^*$ such that $||L||=1$ and

$$d(U, V)=\inf Re\, L(U)-\sup Re\, L(V). \quad \text{Hence}$$

$$d(\hat{U}, \hat{V})=\inf Re\, \hat{L}(\hat{U})-\sup Re\, \hat{L}(\hat{V}) ,$$

$$=\inf Re\, \hat{L}(\hat{U})-\sup Re\, \hat{L}(\tilde{V}) .$$

This gives that the hyperplane

$$H=\{L^{**} \in X^{**}/Re\, \hat{L}(L^{**})=\sup Re\, \hat{L}(\tilde{V})\} ,$$

separates $\hat{U}$, $\tilde{V}$ and hence by Lemmas 3.3 and 3.4 (1),

$$d(\hat{U}, \hat{V})=d(\hat{U}, H)\leq d(\hat{U}, \tilde{V}) .$$

On the other hand, since $\hat{V}\subseteq \tilde{V}$, $d(\hat{U}, \tilde{V})\leq d(\hat{U}, \hat{V})$. This gives $d(\hat{U}, \hat{V})=d(\hat{U}, \tilde{V})$ and the proof is complete.

We conclude this section by simply stating, for the sake of completeness, two more duality theorems for $d(U, V)$, for the case when the promimal points $\bar{u} \in U$, $\bar{v} \in V$ exist. For the first of these theorems, we assume that U, V are contained in a finite dimensional subspace of X. The proof of this theorem essentially follows from Theorem 5.4 and arguments similar to those in Singer [18] (Corollary 1.1, page 173). The proof of the second theorem employs the first theorem. We omit the details.

Theorem 7.8. Let U, V be convex subsets of X such that $d(U, V)>0$ and $\dim [U\cup V]=n$. If the proximal points $\bar{u} \in U$, $\bar{v} \in V$ exist, then there exist m functionals $L_i \in \mathscr{E}(\mathscr{M}_{(\bar{u},\bar{v})})$, $i=1, 2, \cdots, m$, where $m\leq n$ if the scalars are real, $m\leq 2n-1$ if the scalars are complex, such that

$$d(U, V)=||\bar{u}-\bar{v}||=\min_{u \in U, v \in V}\ \max_{1\leq i\leq m} |L_i(u-v)| .$$

Theorem 7.9. Let U, V be convex subsets of X such that $d(U, V)>0$. If the proximal points $\bar{u} \in U$, $\bar{v} \in V$ exist, then

$$d(U, V)=\max_{L \in \mathscr{E}(\mathscr{M}_{(\bar{u},\bar{v})})} |L(\bar{u}-\bar{v})|$$

$$=\min_{u \in U,\, v \in V}\ \max_{L \in \mathscr{E}(\mathscr{M}_{(\bar{u},\bar{v})})} |L(u-v)| .$$

(Here $\mathscr{E}(\mathscr{M}_{(\bar{u},\bar{v})})$ denotes the same set as in Section 5).

REFERENCES

[1] E. W. Cheney: Introduction to Approximation Theory, McGraw-Hill, New York (1966).

[2] E. W. Cheney and A. A. Goldstein: Proximity maps for convex sets, Proc. Amer. Math. Soc. 10 (1959), pp. 448-450.

[3] E. W. Cheney and A. A. Goldstein: Tchebycheff approximation in locally convex spaces, Bull. Amer. Math. Soc., 68 (1962), pp. 449-459.

[4] F. R. Deutsch and P. H. Maserick: Applications of the Hahn-Banach theorem in Approximation Theory, SIAM Review 9 (1967), 516-530.

[5] N. Dunford and J. T. Schwartz: Linear operators—I, Interscience New York, 1958.

[6] N. V. Efimov and S. B. Stechkin: Chebyshev sets in Banach spaces (Russian), Dokl. Akad. Nauk SSSR 121 (1959), pp. 582-585.

[7] A. L. Garkavi: Duality theorems for approximation by elements of convex sets (Russian), Uspehi Mat. Nauk, 16 (1961), pp. 141-145.

[8] A. L. Garkavi: On a criterion for an element of best approximation, (Russian), Sibirsk. Mat. Z., 5 (1964), pp. 472-476.

[9] S. J. Havinson: Approximation by elements of convex sets, (Russian) Soviet. Math. Dokl. 8 (1967), pp. 98-101.

[10] A. N. Kolmogorov: A remark concerning the polynomials of P. L. Tchebycheff which deviate the least from a given function (Russian), Uspehi Math. Nauk 3, (1948), pp. 216-221.

[11] B. R. Kripke and T. J. Rivlin: Approximation in the metric of $L'(X, \mu)$, Trans. Amer. Math. Soc., 119 (1965), pp. 101-122.

[12] V. N. Nikolskii: Best approximation by elements of convex sets in normed linear spaces (Russian), Kalimin—Gos. Ped. Inst. Usen. Pa p. 29 (1963), pp. 85-119.

[13] I. Singer: Caracterisation des éléments de meilleure approximation dans un espace de Banach quelconque, Acta Sci. Math. (Szeged), 17 (1956), pp. 181-189.

[14] I. Singer: On a theorem of V. Pták concerning best approximation of continuous functions in the metric $\int_a^b |x(t)|dt$, Czechoslovak Math. J., 10 (1960), pp. 425-431.

[15] I. Singer: Sur l'extension des functionelles line'aires, Rev. Roumaine Math. Pures Appl., 1 (1956), pp. 1-8.

[16] I. Singer: On the extension of continuous linear functionals and best approximation in normed linear spaces, Math. Ann., 159 (1965), pp. 344-355.

[17] I. Singer: Extremal points, Choquet boundary and best approximation, Rev. Roum. Math. Pures et Appl., 11, 1173-1185 (1966).

[18] I. Singer: Best approximation in normed linear spaces by elements of linear subspaces, Springer-Verlag, Berlin, Heidelberg 1970.

Department of Mathematics,
Indian Institute of Technology,
Powai, Bombay-76, INDIA.