GENERALIZED CONTRACTIONS

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0. Introduction. Let T be a self-mapping on a non-empty complete metric space (X, d). T is a contraction if there exists k in $(0, 1)$ such that

$$d(T(x), T(y)) \leq kd(x, y) , \qquad x, y \in X , \tag{1}$$

The well-known Banach contraction mapping theorem says that every contraction on X has a unique fixed point. The idea of the proof is to show that for any x in X, the sequence $\{T^n(x)\}$ of iterates of x under T is Cauchy and its limit is the unique fixed point of T. A lot of generalizations were obtained by using the above idea with a weaker hypothesis. Among these generalizations, a number of authors (e.g. [3], [4], [7]) assume a condition on T so that, as a consequence, there exists $k \in (0, 1)$ such that

$$d(T(x), T^2(x)) \leq kd(x, T(x)) , \qquad x \in X . \tag{2}$$

This condition together with the continuity of T will insure that T has a fixed point. It is quite obvious that a contraction on X must be continuous and satisfy (2). Let us think T as a motion on X. Let $x \in X$. Then in the first step (from x to $T(x)$), x travels a distance of $d(x, T(x))$; in the second step it travels a distance of $d(T(x), T^2(x))$. (2) requires that the distance that x travels in any step must be equal or less than the preceding step multiplied by a constant less than 1. This condition is too restrictive in the sense that practically, we can observe x walking $(n+1)$ steps (given n). Naturally, we would be able to compare the $(n+1)^{th}$ step with its preceding n steps.

1. Main results.

Theorem 1. Let (X, d) be a metric space. Let T be a self-mapping on X. Suppose that there exist non-negative real numbers $a_1, a_2, \cdots, a_n$ such that

(a) $a_1 + a_2 + \cdots + a_n < 1$,

(b) $d(T^{n+1}(x), T^n(x)) \leq \sum_{i=1}^{n} a_i d(T^i(x), T^{i-1}(x))$, $\qquad x \in X$.

Then $\{T^m(x)\}$ is Cauchy for every x in X.

Proof. Let $x \in X$. By (b),

$$d(T^{n+2}(x), T^{n+1}(x)) \leq \sum_{i=1}^{n} a_i d(T^{i+1}(x), T^i(x)) = \sum_{i=1}^{n-1} a_i d(T^{i+1}(x), T^i(x)) + a_n d(T^{n+1}(x), T^n(x)) ,$$

and

$$d(T^{n+1}(x), T^n(x)) \leq \sum_{i=1}^{n} a_i d(T^i(x), T^{i-1}(x)) .$$

So

$$d(T^{n+2}(x), T^{n+1}(x)) \leq \sum_{i=2}^{n} (a_n a_i + a_{i-1}) d(T^i(x), T^{i-1}(x)) + a_n a_1 d(T(x), x) . \quad (3)$$

Let $i \in \{1, 2, \cdots, n\}$,

$$b_{i,i} = 1 , \quad (4)$$

$$b_{i,j} = 0 , \quad j \neq i , \quad j < n+1 ; \quad (5)$$

$$b_{i,p+1} = \sum_{j=1}^{n} a_j b_{i,j+p-n} , \quad p \geq n . \quad (6)$$

Then obviously, for $p = 1, 2, \cdots, n, n+1$,

$$d(T^p(x), T^{p-1}(x)) \leq \sum_{i=1}^{n} b_{i,p} d(T^i(x), T^{i-1}(x)) . \quad (7)$$

From (3)-(6), (7) holds for $p = n+2$. By induction, suppose that (7) is true for $p \leq k$ $(k \geq n+2)$. Then by (b) and the induction hypothesis,

$$\begin{aligned} d(T^{k+1}(x), T^k(x)) &\leq \sum_{j=1}^{n} a_j d(T^{j+k-n}(x), T^{j+k-n-1}(x)) \\ &\leq \sum_{j=1}^{n} a_j \sum_{i=1}^{n} b_{i,j+k-n} d(T^i(x), T^{i-1}(x)) \\ &= \sum_{i=1}^{n} (\sum_{j=1}^{n} a_j b_{i,j+k-n}) d(T^i(x), T^{i-1}(x)) \\ &= \sum_{i=1}^{n} b_{i,k+1} d(T^i(x), T^{i-1}(x)) . \end{aligned}$$

So

$$d(T^j(x), T^{j-1}(x)) \leq \sum_{i=1}^{n} b_{i,j} d(T^i(x), T^{i-1}(x)) , \quad j = 1, 2, \cdots . \quad (8)$$

From (4)-(6), we have

$$b_{i,n+k+1} < a , \quad k = 0, 1, 2, \cdots, n-1 .$$

where $a = \sum_{i=1}^{n} a_i$. So

$$\begin{aligned} b_{i,2n+k+1} &= \sum_{j=1}^{n} a_j b_{i,j+n+k} \\ &\leq \sum_{j=1}^{n} a_j a \\ &= a^2 , \quad k = 0, 1, 2, \cdots, n-1 . \end{aligned}$$

By induction,

$$b_{i,mn+k+1} \leq a^m ; \quad m = 1, 2, \cdots ; \quad k = 0, 1, 2, \cdots, n-1 .$$

So for $i = 1, 2, \cdots, n$,

$$\begin{aligned} \sum_{j=n+1}^{\infty} b_{i,j} &= \sum_{m=1}^{\infty} (\sum_{k=0}^{n-1} b_{i,mn+k+1}) \\ &\leq \sum_{m=1}^{\infty} (n a^m) \\ &= n \sum_{m=1}^{\infty} a^m . \end{aligned} \quad (9)$$

From (9) and (a), $\sum_{j=1}^{\infty} b_{i,j}$ is convergent for each $i=1, 2, \cdots, n$. So

$$\sum_{i=1}^{n} \left(\sum_{j=1}^{\infty} b_{i,j}\right) d(T^{i}(x), T^{i-1}(x)) < \infty . \tag{10}$$

Since the convergence of a convergent series of non-negative real numbers does not depend on the order of its terms, we have from (8) and (10)

$$\sum_{j=1}^{\infty} d(T^{j}(x), T^{j-1}(x)) < \infty . \tag{11}$$

Let $\varepsilon > 0$. Then there exists a positive integer $n(\varepsilon)$ such that

$$\sum_{j=n(\varepsilon)}^{\infty} d(T^{j}(x), T^{j-1}(x)) < \varepsilon .$$

For $p > q > n(\varepsilon)$,

$$\begin{aligned} d(T^{p}(x), T^{q}(x)) &\leq \sum_{j=q+1}^{p} d(T^{j}(x), T^{j-1}(x)) \\ &\leq \sum_{j=n(\varepsilon)}^{\infty} d(T^{j}(x), T^{j-1}(x)) < \varepsilon . \end{aligned}$$

So $\{T^{m}(x)\}$ is Cauchy.

Theorem 2. Let (X, d) be a non-empty metric space. Let T be a self-mapping on X. Suppose that there exist non-negative real numbers s, t, p_i, q_i, r_i, $i=1, 2, \cdots, n$, such that

(a) $a \equiv s+t+\sum_{i=1}^{n}(p_i+q_i+r_i) < 1$,

(b) for any x, y in X,

$$\begin{aligned} d(T^{n}(x), T^{n}(y)) \leq & \sum_{i=1}^{n} p_i d(T^{i}(x), T^{i-1}(x)) + \sum_{i=1}^{n} q_i d(T^{i}(y), T^{i-1}(y)) \\ & + \sum_{i=1}^{n} r_i d(T^{i-1}(x), T^{i-1}(y)) + s d(T^{n-1}(x), T^{n}(y)) \\ & + t d(T^{n}(x), T^{n-1}(y)) . \end{aligned}$$

Then for any z in X, $\{T^{m}(z)\}$ is Cauchy.

Proof. Let $z \in X$, $x=z$, $y=T(z)$. From (b),

$$\begin{aligned} d(T^{n}(z), T^{n+1}(z)) \leq & \sum_{i=1}^{n} p_i d(T^{i}(z), T^{i-1}(z)) \\ & + \sum_{i=1}^{n} q_i d(T^{i+1}(z), T^{i}(z)) \\ & + \sum_{i=1}^{n} r_i d(T^{i-1}(z), T^{i}(z)) \\ & + s d(T^{n-1}(z), T^{n+1}(z)) . \end{aligned} \tag{12}$$

Since

$$d(T^{n-1}(z), T^{n+1}(z)) \leq d(T^{n-1}(z), T^{n}(z)) + d(T^{n}(z), T^{n+1}(z)) ,$$

from (12),

$$\begin{aligned} (1-s-q_n) d(T^{n}(z), T^{n+1}(z)) \leq & (p_1+r_1) d(T(z), z)) \\ & + \sum_{i=2}^{n-1} (p_i+q_{i-1}+r_i) d(T^{i}(z), T^{i-1}(z)) \\ & + (p_n+q_{n-1}+r_n+s) d(T^{n}(z), T^{n-1}(z)) . \end{aligned} \tag{13}$$

Let $x=T(z)$, $y=z$. Then from (b),

$$(1-t-p_n)d(T^{n+1}(z),\ T^n(z))\leq(q_1+r_1)d(T(z),\ z)$$
$$+\textstyle\sum_{i=2}^{n-1}(q_i+p_{i-1}+r_i)d(T^i(z),\ T^{i-1}(z))$$
$$+(q_n+p_{n-1}+r_n+t)d(T^n(z),\ T^{n-1}(z))\ . \tag{14}$$

Simplifying (13)+(14), we have

$$(2-s-t-p_n-q_n)d(T^{n+1}(z),\ T^n(z))\leq(p_1+q_1+2r_1)d(T(z),\ z)$$
$$+\textstyle\sum_{i=2}^{n-1}(p_i+q_i+p_{i-1}+q_{i-1}+2r_i)d(T^i(z),\ T^{i-1}(z))$$
$$+(p_n+q_n+p_{n-1}q_{n-1}+2r_n+s+t)d(T^n(z),\ T^{n-1}(z))\ .$$

Let

$$\alpha_1=\frac{p_1+q_1+2r}{2-s-t-p_n-q_n}\ , \tag{15}$$

$$\alpha_i=\frac{p_i+q_i+p_{i-1}+q_{i-1}+2r_i}{2-s-t-p_n-q_n}\ ,\qquad i=2,\ 3,\ \cdots,\ n-1\ , \tag{16}$$

$$\alpha_n=\frac{p_n+q_n+p_{n-1}+q_{n-1}+s+t+2r_n}{2-s-t+p_n-q_n}\ . \tag{17}$$

Then $\alpha_1,\ \alpha_2,\ \cdots,\ \alpha_n\in[0,\ \infty)$ and

$$d(T^{n+1}(z),\ T^n(z))\leq\textstyle\sum_{i=1}^{n}\alpha_i d(T^i(z),\ T^{i-1}(z))\ .$$

Now

$$\textstyle\sum_{i=1}^{n}\alpha_i=\frac{2a-s-t-p_n-q_n}{2-s-t-p_n-q_n}<1\ .$$

So by Theorem 1, $\{T^m(z)\}$ is Cauchy.

2. Fixed point theorems.

Theorem 3. Let (X, d) be a non-empty complete metric space. Let $\mathcal{F}$ be a family of commuting self-mappings on X. Suppose that there exists T in $\mathcal{F}$ such that T is continuous and satisfies the conditions of Theorem 2. Then T has a unique fixed point x_0. Hence x_0 is the common fixed point of $\mathcal{F}$.

Proof. Let $z\in X$. By Theorem 2, $\{T^n(z)\}$ is Cauchy. By completeness of (X, d), $\{T^n(z)\}$ converges to some x in X. By continuity of T, x is a fixed point of T. Let x, y be fixed points of T. Then by (b) in Theorem 2,

$$d(x,\ y)=d(T^n(x),\ T^n(y))\leq(\textstyle\sum_{i=1}^{n}r_i+s+t)d(x,\ y)\ .$$

Since $\sum_{i=1}^{n}r_i+s+t<1$, $x=y$. So T has a unique fixed point x. Let $S\in\mathcal{F}$. Then

$$S(x)=S(T(x))=T(S(x))\ .$$

So $S(x)$ is a fixed point of T. By uniqueness $S(x)=x$. So x is the common fixed point to $\mathscr{F}$.

If (X, d) in Theorem 3 is replaced by a generalized complete metric space [2], [5], [8], then, to obtain the same conclusion, we need to assume further that for some x, $d(T^i(x), T^{i-1}(x))<\infty$ for n consecutive i's. Also, in practice, the conditions of Theorem 3 can be weakened: if we know $\{T^m(z)\}$ converges to x, then we need only to assume that T is continuous at x. For example, let X be the real line with the usual distance, let T be the self-mapping on X defined by

$$T(x)=\frac{1}{2}x \quad \text{if } x \text{ is irrational},$$

$$T(x)=-\frac{1}{2}x \quad \text{if } x \text{ is rational}.$$

Then T is not contractive since T is continuous only at $x=0$. But T^2 is contractive. So by Theorem 2, $\{T^m(z)\}$ is Cauchy at every z in X. Now $\{T^m(z)\}$ converges to zero and T is continuous at zero. So T has a fixed point at zero. Note that if we apply the Banach contraction mapping to T^2. We will know that $\{T^{2m}(z)\}$ but not $\{T^m(z)\}$ is Cauchy. Sherwood C. Chu and J. B. Diaz [1] noted that "It is of interest to notice that an example of a discontinuous transformation A, with A^2 contracting, can be given even when the metric space R is the set of real numbers". They constructed an example by using Zorn's lemma. We note here that there are quite a few such examples whose construction do not depend on the axiom of choice. The above T is one! In fact, there also exist a lot of continuous self-mapping T on the real line R such that T^2 but not T is a contraction, e.g., if

$$T(x)=-\frac{1}{2}x \quad \text{if} \quad x>0,$$

$$T(x)=-x \quad \text{if} \quad x\leq 0,$$

then T is such a mapping.

Theorem 4. Let (X, d) be a non-empty compact metric space. Let T be a continuous self-mapping on X. Suppose that there exist non-negative real numbers $s, t, p_i, q_i, r_i, i=1, 2, \cdots, n$ such that

(a) $s+t+\sum_{i=1}^n (p_i+q_i+r_i)=1$;

(b) for any distinct x, y in X,

$$\begin{aligned} d(T^n(x), T^n(y)) < \textstyle\sum_{i=1}^n p_i d(T^i(x), T^{i-1}(x)) \\ + \textstyle\sum_{i=1}^n q_i d(T^i(y), T^{i-1}(y)) \\ + \textstyle\sum_{i=1}^n r_i d(T^{i-1}(x), T^{i-1}(y)) \\ + s d(T^{n-1}(x), T^n(y)) + t d(T^n(y), T^{n-1}(x)) . \end{aligned}$$

Then T has a unique fixed point.

Proof. We shall first prove that T has a fixed point. Let $z \in X$. Suppose that $T^{n-1}(z)$ is not a fixed point of T. By (b) and a similar argument as in the proof of Theorem 2_i we obtain

$$\begin{aligned} (2-s-t-p_n-q_n)d(T^{n+1}(z), T^n(z)) < (p_1+q_1+2r_1)d(T(z), z) \\ + \textstyle\sum_{i=2}^{n-1} (p_i+q_i+p_{i-1}+q_{i-1}+2r_i)d(T^i(z), T^{i-1}(z)) \\ + (p_n+q_n+p_{n-1}+q_{n-1}+2r_n+s+t)d(T^n(z), T^{n-1}(z)) . \end{aligned}$$

So $2-s-t-p_n-q_n>0$ (otherwise, from (a), the coefficients of $d(T^{j-1}(z), T^{j-1}(z))$ in the above inequality is equal to 0 for each $j=1, 2, \cdots, n$ whence $0<0$, a contradiction). Therefore $\alpha_1, \alpha_2, \cdots, \alpha_n$ defined by (15)-(17) are real numbers. Now we have

$$\alpha_1+\alpha_2+ \cdots +\alpha_n=1 ,$$

and

$$\begin{aligned} d(T^{n+1}(z), T^n(z)) < \textstyle\sum_{i=1}^n \alpha_i d(T^i(z), T^{i-1}(z)) \\ \le \max\{d(T^i(z), T^{i-1}(z)) : \ i=1, 2, \cdots, n\} , \end{aligned} \tag{18}$$

where $z \in X$, $T^n(z) \neq T^{n-1}(z)$. Consider the function ϕ on X defined by

$$\phi(x)=\max\{d(T^i(x), T^{i-1}(x)) : \ i=1, 2, \cdots, n\} , \quad x \in X .$$

Since T is continuous, ϕ is continuous. So ϕ takes its minimum value at some point w in X. We claim that $T^{m-1}(w)$ is a fixed point of T for some $m \ge n$. For if not, then applying (18) to $T^{n+j-1}(w)$, $j=0, 1, 2, \cdots, n-1$, we obtain

$$\phi(T^n(w)) < \phi(w) ,$$

a contradiction to the choice of w. Now let x, y be fixed points of T. If $x \neq y$, then by (b),

$$d(x, y)=d(T^n(x), T^n(y)) < (\textstyle\sum_{i=1}^n r_i+s+t)d(x, y) \le d(x, y) ,$$

a contradiction. So T has a unique fixed point.

Let (X, d) be a non-empty complete metric space. Let T be a continuous self-mapping on X which satisfies the conditions of Theorem 2. One wonders if there is an equivalent complete metric for X under which T becomes a contraction. The following result gives an affirmative answer.

Theorem 5. Let (X, d) be a non-empty complete metric space. Let T be a continuous self-mapping on X which satisfies the conditions of Theorem 2. Then for some k in $(0, 1)$, there exists a complete metric d_k for X such that d_k is topologically equivalent to d and

$$d_k(T(x), T(y)) \leq k d_k(x, y) , \qquad x, y \in X .$$

Proof. By Theorem 3, T has a unique fixed point x_0. Since T is continuous, by a result of P. R. Meyer [6, Theorem 1], we need only to prove that $\{T^p(V)\}$ converges to x_0 for some neighborhood V of x_0. Let

$$V = \{x \in X : d(T^{i-1}(x), x_0) < 1 , \quad i = 1, 2, \cdots, n\} .$$

Then $x_0 \in V$. By continuity of T, V is a neighborhood of x_0. Let $\varepsilon > 0$. It suffices to prove that $T^p(V) \subset \{x \in X : d(x, x_0) < \varepsilon\}$ for large p's. Let $z \in X$. By (b) in Theorem 2 (with $x = z$, $y = x_0$),

$$\begin{aligned} d(T^n(z), x_0) &= d(T^n(z), T^n(x_0)) \\ &\leq \textstyle\sum_{i=1}^{n} p_i d(T^i(z), T^{i-1}(z)) + \sum_{i=1}^{n} r_i d(T^{i-1}(z), x_0) \\ &\quad + s d(T^{n-1}(z), x_0) + t d(T^n(z), x_0) . \end{aligned} \tag{19}$$

Since

$$d(T^i(z), T^{i-1}(z)) \leq d(T^i(z), x_0) + d(T^{i-1}(z), x_0) , \qquad i = 1, 2, \cdots, n ,$$

from (19), we obtain

$$\begin{aligned} (1 - t - p_n) d(T^n(z), x_0) &\leq \textstyle\sum_{i=2}^{n-1} (p_{i-1} + r_i + p_i) d(T^{i-1}(z), x_0) \\ &\quad + (r_1 + p_1) d(z, x_0) + (p_{n-1} + r_n + p_n + s) d(T^{n-1}(z), x_0) . \end{aligned} \tag{20}$$

Similarly, be letting $x = x_0$, $y = z$, we have

$$\begin{aligned} (1 - s - q_n) d(T^n(z), x_0) &\leq \textstyle\sum_{i=2}^{n-1} (q_{i-1} + r_i + q_i) d(T^{i-1}(z), x_0) \\ &\quad + (r_1 + q_1) d(z, x_0) + (q_{n-1} + r_n + q_n + t) d(T^{n-1}(z), x_0) . \end{aligned} \tag{21}$$

From (20) and (21), we have

$$d(T^n(z), x_0) \leq \textstyle\sum_{i=1}^{n} \alpha_i d(T^{i-1}(z), x_0) , \qquad z \in X ,$$

where α_i's were defined by (15), (16) and (17). By repeating a similar argument as in the proof of Theorem 1, we obtain

$$d(T^p(z), x_0) \leq \textstyle\sum_{i=1}^{n} b_{i,p} d(T^{i-1}(z), x_0) , \qquad z \in X , \tag{22}$$

where $b_{i,p}$'s were defined by (4), (5) and (6). Since $\{b_{i,p}\}$ converges to zero for each $i = 1, 2, \cdots, n$, there exists a positive integer $n(\varepsilon)$ such that

$$b_{i,p} < \varepsilon \quad \text{for all} \quad i = 1, 2, \cdots, n ; \; p \geq n(\varepsilon) .$$

So from (22),

$$d(T^p(z), x_0) < \varepsilon \quad \text{for all} \quad z \in V, \quad p \geq n(\varepsilon),$$

i.e. $T^p(V) \subset \{x \in X : d(x, x_0) < \varepsilon\}$ for all $p \geq n(\varepsilon)$.

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