A VERSION OF BIRKHOFF-FRINK'S THEOREM IN THE ABSTRACT GEOMETRY

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The following theorem of Birkhoff-Frink is a well-known important result for the lattice of subalgebras of an abstract algebra:

Theorem. A lattice L is isomorphic with a subalgebra-lattice if and only if L is complete, meet-continuous, and every element of L is a join of join-inaccessible elements. ([1] and also [2], Thm. 9, p. 188)

It is intended in this note to give a different version of this theorem so that some results of abstract geometry can also be subordinated to it.

1. It is known that one can identify every hull-operation in a complete lattice V with a hull-system which is a complete $\cap$-subband ($\cap$-Teilbund) of V. ([3] and also [4] Satz 6.5 p. 32)

When V is the complete lattice $P(m)$ of all subsets of a set m, then the hull-operation is a closure operation and to each such operation is associated a closure property.

A closure property Φ associated with a closure operation $X \rightarrow \bar{X}$ on the subsets X of the set m is said to be *finitary* if the condition $X \in \Phi$ is equivalent to the condition that $K \subset X$ and, K finite imply $\bar{K} \subset X$. ([2] p. 185)

The complete $\cap$-subbands of $P(m)$ which correspond to closure operation whose associated closure properties are finitary are characterized by the following:

Theorem 1. Let A be a complete $\cap$-subband of $P(m)$. Then the following two conditions are equivalent: (See [2], Lemma 1 p. 186).

a) There is a closure operation on $P(m)$, for which the associated closure property is finitary and the subset of m belongs to A if and only if it is closed under the closure operation.

b) For each directed set M contained in A, the set union $\cup M$ is also contained in A.

The proof of this theorem can be obtained by modifying the proof of the corresponding theorem in the book of Hermes ([4] Satz 7.2, p. 35 or see the appendix to this note).

As a corollary to Theorem 1, one gets the following:

Corollary 1. For a complete $\cap$-subband A satisfying a) and a subset $X \subset m$, $\bar{X}$ is the union of all $\bar{Y}$ with finite $Y \subset X$.

Proof. Consider the family $\bar{Y}$ for all finite subset Y of X. Then $\bar{Y} \subset \bar{X}$ and $\cup \bar{Y} \subset \bar{X}$. Now this family is obviously directed, so $\cup \bar{Y}$ is closed and $\cup \bar{Y} \supset X$. Hence $\cup \bar{Y} \supset \bar{X}$ and one gets $\bar{X} = \cup \bar{Y}$.

2. There are many interesting examples of complete $\cap$-subbands of $P(m)$ which belong to this category. For examples, the lattice of subalgebras of an algebra $[S, F]$ ([2] p. 132 and p. 185, and Theorem 6, p. 186), the lattice of subgeometries of an abstract geometry with finitary operation in the sense of Maeda [5], and the lattice of flats of (merely finitary) geometry as defined by Jonsson [6].

A (merely finitary) *geometry* (as defined by Jonsson) is an ordered pair $\langle S, C \rangle$ consisting of a set S and a function C which associates with every subset X of S another subset $C(X)$ (which is also denoted as $\bar{X}$ in the sequel) of S in such a way that the following conditions are satisfied:

(i) $X \subseteq C(X) = C(C(X))$ for every subset X of S,

(ii) $C(p) = p$ for every $p \in S$,

(iii) $C(\phi) = \phi$, where ϕ is the empty set,

(iv) For every subset X of S, $C(X)$ is the union of of all sets of the form $C(Y)$ with Y a finite subset of X.

It is obvious that $X \to C(X)$ is a closure operation in $P(m)$ and (iv) implies the "finitary" property of the associated closure property. Suppose that Y is finite and $C(Y) \subset X$. Then, since $C(X)$ is the union of $C(Y)$, $C(X) \subset X$, hence $X = C(X)$.

Now the abstract geometry with finitary operation in the sense of Maeda is defined as follows:

Let G be a set of points. If for any finite points $p_1, \cdots, p_n$ of G, there exists a subset $p_1 + \cdots + p_n$ (for any i such that $1 \leqslant i \leqslant n$, denoting $p_i + \cdots + p_n = p_i + p_i$ when $n = i$) of G containing p_i which satisfies

(1°) $p_1 = p_2$ implies $p_1 + p_2 + \cdots + p_n = p_2 + \cdots + p_n$,

(2°) for any permutation $p_{i_1}, \cdots, p_{i_n}$ of $p_1, \cdots, p_n$

$$p_1 + \cdots + p_n = p_{i_1} + \cdots + p_{i_n}$$

(3°) $q_i \in p_1^{(i)} + \cdots + p_{n_i}^{(i)} (i = 1, \cdots, m)$ imply

$$q_1 + \cdots + q_m \subseteq p_1^{(1)} + \cdots + p_{n_1}^{(1)} + p_1^{(2)} + \cdots \quad \cdots + p_1^{(m)} + \cdots + p_{n_m}^{(m)},$$

Then G is called an abstract geometry with finitary operations. A subset H of G is called a subgeometry of G if $p_1, \cdots, p_n \in H$ implies $p_1 + \cdots + p_n \subseteq H$. It follows that $p_1 + \cdots + p_n$ is a subgeometry.

In an abstract geometry G with finite operation, one can define a closure operation as follows: Let B be any subset of G, then define $\bar{B}$ to be the smallest subgeometry containing B. Then it is obvious that $B \subseteq \bar{B}$, $\bar{\bar{B}} = \bar{B}$ and that $B \subseteq C$ implies $\bar{B} \subseteq \bar{C}$. Thus $B \to \bar{B}$ is a closure operation. Its associated closure property is finitary, since $B = \bar{B}$ means that B is a subgeometry, and B is a subgeometry if and only if $K = \{p_1, \cdots, p_n\} \subset B$ implies $\bar{K} = p_1 + \cdots + p_n \subseteq B$.

Conversely a set m with a closure operation whose associated closure property is finitary is an abstract geometry with finitary operation in the sense of Maeda (compare with [2] Theroem 6, p. 186).

For $K = \{p_1, \cdots, p_n\} \subset G$, define $p_1 + \cdots + p_n = \bar{K} \subseteq G$. Then since $K \subseteq \bar{K}$, $p_1, \cdots, p_n \in p_1 + \cdots + p_n$.

(1°) If $p_1 = p_2$ and $K_1 = \{p_1, p_2, \cdots, p_n\}$, $K_2 = \{p_2, \cdots, p_n\}$, then $K_1 = K_2$ (as sets) which implies $\bar{K}_1 = \bar{K}_2$, that is $p_1 + p_2 + \cdots + p_n = p_2 + \cdots + p_n$.

(2°) Let $K = \{p_1, \cdots, p_n\}$ and $K_\pi = \{p_{i_1} \cdots p_{i_n}\}$, where $\pi : (1 \cdots n) \to (i_1 \cdots i_n)$ is a permutation, then $K = K_\pi$ (as sets) and $\bar{K} = \bar{K}_\pi$, that is $p_1 + \cdots + p_n = p_{i_1} + \cdots + p_{i_n}$.

(3°) Let $q_i \in \bar{K}_i = p_1^{(i)} + \cdots + p_{n_i}^{(i)}$, $(i = 1, \cdots, m)$, where $K_i = \{p_1^{(i)}, \cdots, p_{n_i}^{(i)}\}$. Then $K_1, \cdots, K_m \subseteq K = \{p_1^{(1)}, \cdots, p_{n_1}^{(1)}, \cdots, p_1^{(m)}, \cdots, p_{n_m}^{(m)}\}$ and $\bar{K}_1, \cdots, \bar{K}_m \subseteq \bar{K}$. Hence $Q = \{q_1, \cdots, q_m\} \subset \bar{K}$ which implies $\bar{Q} \subset \bar{K}$, that is, $q_1 + \cdots + q_m \subseteq p_1^{(1)} + \cdots + p_{n_1}^{(1)} + \cdots + p_1^{(m)} + \cdots + p_{n_m}^{(m)}$.

A subgeometry is defined by the condition that $\{p_1, \cdots, p_n\} \subset A$ implies $p_1 + \cdots + p_n \subset A$. This means that a subset is closed under the finitary closure property if and only if it is a subgeometry.

It is understood in the above definition of abstract geometry with finitary operation that $\bar{p} = p$ is not assumed.

3. The above Theorem 1 gives the characterizing property of the complete $\cap$-subband of $P(m)$, which corresponds to a closure operation of $P(m)$ whose associated closure property is finitary. Now one may propose to characterize a lattice which is isomorphic to such a complete $\cap$-subband of a $P(m)$. For the answer to this question one will come up to the following version of Birkhoff-Frink's theorem (See [2], Theorem 9, p. 188).

Theorem 2. The following three conditions for a lattice V are necessary and sufficient for the existence of a set m and a hull-operation on $P(m)$ whose associated closure property is finitary so that V is isomorphic to the lattice A of all subsets of m which are closed under the hull-operation.

(1) V is complete,

(2) $x(\Sigma y_\rho)=\Sigma xy_\rho$ for each $x\in V$ and every directed set $\{y_\rho\}$ of elements of V,

(3) every element of V is the join of a set of inaccessible elements.

In the statement, an inaccessible element is defined as follows: An element a of a poset P is said to be accessible if there is a directed subset A of P such that $a\notin A$ and $\Sigma A=a$. Otherwise a is said to be inaccessible.

Proof of this theorem can be obtained by modifying that contained in the book of Hermes ([4], Theorem 7.4, pp. 37-40 or see appendix).

Apply this theorem to a (merely finitary) geometry, by taking $\bar{p}=p$ into account, one gets the following corollary:

Corollary 2. The lattice of flats $(A=\bar{A})$ of a (merely finitary) geometry $\langle m, c\rangle$ is complete upper-continuous (meet-continuous) and atomistic (i.e. $\bar{A}=\{\text{atom } p|p\leq\Sigma\bar{A}\}$).

The last property follows from the fact shown in the proof of Theorem 2 (see appendix) that every element is the union of $\bar{p}=p$.

Conversely, if a lattice $\mathscr{L}$ is complete, upper-continuous and atomistic, then, since an atom in such a lattice is easily seen to be (join) inaccessible, it is isomorphic, by Theorem 2, to the lattice $\mathscr{L}'$ of closed elements under the closure operation $C': B\to\bar{B}'=\{u|u$ inaccessible and $u\leq\Sigma B\}$ with finitary closure property. (i.e. isomorphic to a lattice of subgeometries of an abstract geometry with finitary operation). But $\langle m'\ c'\rangle$ is not a (merely finitary) geometry, because, for each inaccessible element u, $\bar{u}=u$ does not hold generally ($\bar{\phi}=\{u|u\leq\phi\}$, so $\bar{\phi}=\phi$).

Let m be the set of all atoms in $\mathscr{L}$. For $A\subset m\subset m'$, let $\bar{A}=\{\text{atom } p|p\leq\Sigma A\}$. Then evidently $A\to\bar{A}$ is a closure operation C, and the associated closure property can be shown to be finitary: Since $m\subset m'$ and C' is finitary, so if $p\in\bar{A}$ then $p\in\bar{A}'$ (since $\bar{A}\subset\bar{A}'$) and there are inaccessible elements $u_1, \cdots, u_n\in A$ such that $p\leq u_1+\cdots+u_n$, where $u_1, \cdots, u_n$ are atoms, since they belong to A. Since $\bar{p}=p$ and $\bar{\phi}=\phi$ are obvious from the definition of $\bar{A}$, $\langle m, c\rangle$ is a (merely finitary) geometry.

Now the correspondence $\bar{A}=\{\text{atom } p|p\leq\Sigma\bar{A}\}\to\bar{A}'=\{\text{inaccessible } u|u\leq\Sigma\bar{A}\}$ is one-one, since $\bar{A}'=\bar{B}'$ implies $\Sigma\bar{A}=\Sigma\bar{B}$ (since $\Sigma\bar{A}'=\Sigma\bar{A}$) which in turn implies $\bar{A}=\bar{B}$. It is also onto, because for a given $\bar{A}'$, $\bar{A}=\{p: \text{atom}|p\leq\Sigma\bar{A}'\}\to\bar{A}'$ (since $\Sigma\bar{A}=\Sigma\bar{A}'$) and obviously $\bar{A}\subset\bar{B}$ if and only if $\bar{A}'\subset\bar{B}'$. Thus the lattice of flats of the (merely finitary) geometry $\langle m, c\rangle$ is isomorphic to the lattice of closed elements of $p(m')$ under the closure operation C'.

By this isomorphism and the Theorem 2, one can derive the following theorem proved directly for an abstract geometry with finitary operation by Maeda ([5], Theorems 2.1 and 2.2, p. 93).

Theorem 3. A lattice is geometric if and only if it is complete, upper continuous and atomistic.

In the statement of this theorem, a geometric lattice means the one which is isomorphic to the lattice of all flats of some (merely finitary) geometry.

4. As an application of the Theorem 3, consider the so-called geometry of grade n [7] (or Wille's geometry of grade n [8]) which is defined as follows:

Suppose that in the set S of points, a family of subsets of S, each of which is called a *curve*, and another family of subsets of S, each of which is called a *surface*, are specified such that the following postulates are satisfied:

$P(1)$ $n+1$ distinct points are contained in exactly one curve, and each curve contains at least $n+1$ distinct points.

$P(2)$ $n+2$ distinct points, which are not contained in a curve are contained in exactly one surface, and in each surface there are at least $n+2$ distinct points which are not contained in a curve.

$P(3)$ Along with the $(n+1)$ distinct points contained in a surface, the curve determined by these points is also contained in the surface.

A point set is called a subspace (or a flat) if it contains all the curves and surfaces which are determined according to P(1) and P(2) by $n+1$ distinct or $n+2$ distinct points contained in the set.

Thus a curve and a surface are subspaces and so is also any point set which consists of not greater than n distinct points.

The intersection of all subspaces, which contain a set A is called the closure of A and is denoted by $\bar{A}$ (or $C(A)$).

$P(4)$ The intersection of two surfaces which are contained in the closure of $n+3$ points, will never consist of exactly n distinct points.

The set S of points together with the families of curves and surfaces which satisfy the postulates $P(1)$-$P(4)$ will be called a Wille (incidence) geometry of grade n of the set S.

Wille [7] has proved the following theorem which characterize lattice theoretically the Wille's geometry of grade n.

Theorem 4. A lattice L is isomorphic to the lattice of flats of a Wille's geometry of grade n if and only if the lattice is geometric, semi-modular and for each flat x of rank n in L the interval $[x\ 1]$ is modular, and the interval $[0\ x]$ is distributive.

Part of this theorem follows from Theorem 3 and Theorem 1. In Wille's geometry of grade n, $A \rightarrow \bar{A}$, the least flat containing A, is obviously a closure operation and the associated closure property is finitary, since for a directed family of flats, the set union of these flats is again a flat, so finitary property follows from Theorem 1. Thus by Theorem 3, the lattice of flats of this geometry is complete, upper-continuous and atomistic, so for the proof of the "necessity" part of the theorem, it needs only to show (α) the semi-modularity and further properties of this lattice.

Assume conversely that $\mathscr{L}$ satisfies all the properties given in Theorem 4. It follows, by Theorem 3, that $\mathscr{L}$ is isomorphic to the lattice of flats ($A=\bar{A}$) of the (merely finitary) geometry $\langle m, c\rangle$ where m is the set of all atoms in $\mathscr{L}$ and for any subset $A\subset m$, $C(A)\equiv\bar{A}\equiv\{p:\ \text{atom}\,|\,p\leq\Sigma A\}$.

Since $\mathscr{L}$ is geometric and semi-modular, for any finitely generated elements a, one defines "rank" in the interval $[0\ a]$. If one calls a flat of the (merely finitary) geometry $\langle m, c\rangle$ which corresponds to an element of rank $n+1$, a *curve* and that corresponding to an element of rank $n+1$, a *curve* and that corresponding to an element of rank $n+2$, a *surface*, then it can be easily shown that the geometry $\langle m, c\rangle$ satisfies $P(1)$, $P(2)$ and $P(3)$.

Now call a subset A of m a *subspace* if the surface $\langle p_1, \cdots, p_{n+2}\rangle\subset A$, whenever $n+2$ distinct points $p_1, \cdots, p_{n+2}$ which belong to A. Then, by Theorem 3, for the proof of the "sufficiency" part of the Theorem 4, it needs only to show that (β_1) $P(4)$ holds and that (β_2) every subspace is a closed set of the geometry $\langle m, c\rangle$.

5. If one assumes the results on the lattice-theoretic characterization of projective geometry of infinite dimension ([9] or [4] §§14-16, pp. 75-91), that is, the special case $n=0$ of Theorem 4, the proofs of parts (α) and (β) can be easily obtained as follows:

Let $p_1, \cdots, p_n$ be any n distinct given points. Call each curve and surface containing $\{p_1, \cdots, p_n\}$ respctively a *p-point* and a *p-line*. A p-line and p-point are said to be *incident* if the correspoinding surface contains the corresponding curve set-theoretically. Then one can prove that

($P'1$) Two distinct p-points lie on one and only one p-line.

($P'2$) Let α, α', β, β', γ be p-points. If $\alpha'\ \beta\ \gamma$ are collinear, and α, β', γ are also collinear, then there exists a p-point γ' such that $\alpha\ \beta\ \gamma'$ are collinear, and α'

$\beta' \gamma'$, are also collinear. This corresponds to the case $n=0$ of $P(4)$). Thus the set of p-points and p-lines forms a projective geometry.

For the proof of $(P'1)$, let the two distinct p-points be given by $\langle p_1, \cdots, p_n, q_i\rangle$, $(i=1, 2)$. Since these two p-points are distinct, $q_2 \notin \langle p_1, \cdots, p_n, q_1\rangle$, so $\langle p_1, \cdots, p_n, q_1, q_2\rangle$ is a surface which contains both given curves. Thus there is a p-line which contains the given two p-points. Any surface which contains both these two curves contains $\{p_1, \cdots, p_n, q_1, q_2\}$, so by $P(2)$, it coincides with $\langle p_1, \cdots, p_n, q_1, q_2\rangle$.

For the proof of $(P'2)$, suppose that $\alpha, \alpha', \beta, \beta', \gamma$ are represented by $\langle p_1, \cdots, p_n, a\rangle$, $\langle p_1, \cdots, p_n, a'\rangle$, $\cdots$, $\langle p_1, \cdots, p_n, c\rangle$ respectively. From the collinearity, it follows that $\langle p_1, \cdots, p_n, a', b, c\rangle$ and $\langle p_1, \cdots, p_n, a, b', c\rangle$ are surfaces, thus $\langle p_1, \cdots, p_n, a', b, c\rangle$, $\langle p_1, \cdots, p_n, a, b' c\rangle \subset \langle p_1, \cdots, p_n, a, b, c\rangle$. Hence, $\langle p_1, \cdots, p_n, a', b'\rangle$, $\langle p_1, \cdots, p_n, a, b\rangle \subset \langle p_1, \cdots, p_n, a, b, c\rangle$. Now if both $\langle p_1, \cdots, p_n, a, b\rangle$ and $\langle p_1, \cdots, p_n, a', b'\rangle$ are sarfaces, then, by $P(4)$, there exists a point c' such that $c' \in \langle p_1, \cdots, p_n, a, b\rangle \cap \langle p_1, \cdots, p_n, a', b'\rangle$, so both $\langle p_1, \cdots, p_n, a, b, c'\rangle$ and $\langle p_1, \cdots, p_n, a', b', c'\rangle$ are surfaces.

If $\langle p_1, \cdots, p_n, a, b\rangle$ is a curve, but $\langle p_1, \cdots, p_n, a', b'\rangle$ is a surface, then there is a point c' in $\langle p_1, \cdots, p_n, a', b'\rangle$ such that $c' \notin \langle p_1, \cdots, p_n, a, b\rangle$. Then $\langle p_1, \cdots, p_n, a, b, c'\rangle$ and $\langle p_1, \cdots, p_n, a', b', c'\rangle$ are both surfaces.

If both $\langle p_1, \cdots, p_n, a, b\rangle$ and $\langle p_1, \cdots, p_n, a', b'\rangle$ are curves, then there is a surface containing these two curves (by $(P1)$). Let c' be any point other than p_i $(i=1, \cdots, n)$ in this surface, then $\langle p_1, \cdots, p_n, a, b, c'\rangle$ and $\langle p_1, \cdots, p_n, a', b', c'\rangle$ are at most surfaces.

Let A be any subspace (in the Wille geometry) which contains the given n distinct points $p_1, \cdots, p_n$. It is easily seen that the set of p-points which correspond to curves contained in A and contain the given n points is a p-flat in the projective geometry.

Let ϕ be any p-flat of the corresponding projective geometry, and let A be the set union of all the curves which correspond to p-points contained in ϕ. It can be shown that A *is a subspace of the Wille geometry* as follows: It is obvious that A has the property $\langle p_1, \cdots, p_n, q_1, q_2\rangle \subset A$ for any two distinct points q_1, q_2 in A. Thns, the above claim from the following Lemma 1 proved by Wille:

Lemma 1. Let A be a subset of m, which contains at least n distinct points $p_1, \cdots, p_n$. If A contains the surface $\langle p_1, \cdots, p_n, q_1, q_2\rangle$ along with any two points q_1, q_2 of A, then A is a subspace.

For the proof of the Lemma 1, one proves first that $\langle p_1, \cdots, p_n, q_{n+1}, q_{n+2}\rangle \subset A$ for fixed n distinct points $p_1, \cdots, p_n$ of A and arbitrary q_{n+1}, q_{n+2} in A implies $\langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{n+2}\rangle \subset A$ for fixed n distinct points $p_1, \cdots, p_{n-1}, q_n$ $(\neq p_n)$ of A

and arbitrary q_{n+1}, q_{n+2} in A. If this is shown, then one replaces p_{n-1} by q_{n-1}, p_{n-2} by $q_{n-2}, \cdots$, one by one, and finally reaches to $\langle q_1, \cdots, q_n, q_{n+1}, q_{n+2}\rangle \subset A$.

Let $q \in \langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{n+2}\rangle$ be any point. One can assume that $\langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{+2}\rangle$ is a surface, since otherwise $\langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{n+2}\rangle = \langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}\rangle \subset \langle p_1, \cdots, p_{n-1}, p_n, q_n, q_{n+1}\rangle \subset A$. One can also assume that $q \notin \langle p_1, \cdots, p_{n-1}, q_{n+1}, q_{n+2}\rangle$, since otherwise $q \in \langle p_1, \cdots, p_{n-1}, p_n, q_{n+1}, q_{+2}\rangle \subset A$. Thus, $\langle p_1, \cdots, p_{n-1}, q, q_{n+1}, q_{n+2}\rangle = \langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{n+2}\rangle$, and $q_n \in \langle p_1, \cdots, p_{n-1}, p_n, q, q_{n+1}, q_{n+2}\rangle$. If either q_{n+1} or q_{n+2} coincides with p_n, then it is obvious that $q \in A$, so one can assume that $\langle p_1, \cdots, p_{n-1}, p_n, q_{n+1}, q_{n+2}\rangle$ is a surface. On the other hand, one can also assume that $\langle p_1, \cdots, p_{n-1}, q_n, p_n, q\rangle$ is a surface, since otherwise $q \in \langle p_1, \cdots, p_{n-1}, p_n, q_n\rangle \subset \langle p_1, \cdots, p_{n-1}, q_n, p_n, q_{n+1}\rangle \subset A$. Then, by $P(4)$ there is a point $r \in \langle p_1, \cdots, p_{n-1}, p_n, q_{n+1}, q_{n+2}\rangle \cap \langle p_1, \cdots, p_{n-1}, p_n, q, q_n\rangle$ which is distinct from $p_1, \cdots, p_n$. Thus $r \in A$. If $r \in \langle p_1, \cdots, p_{n-1}, p_n, q_n\rangle$ then $q_n \in \langle p_1, \cdots, p_{n-1}, p_n, r\rangle \subset \langle p_1, \cdots, p_n, q_{n+1}, q_{n+2}\rangle$ and $q \in \langle p_1, \cdots, p_{n-1}, q_n, q_{n+1}, q_{n+2}\rangle = \langle p_1, \cdots, p_n, q_{n+1}, q_{n+2}\rangle \subset A$. If $r \notin \langle p_1, \cdots, p_n, q_n\rangle$ then $\langle p_1, \cdots, p_{n-1}, p_n, q_n, r\rangle$ is a surface, so $\langle p_1, \cdots, p_n, q_n, r\rangle = \langle p_1, \cdots, p_n, q_n, q\rangle \ni q$.

Remark. The proof of this lemma is motivated by the case $n=1$ in three dimensional space with the following configuration:

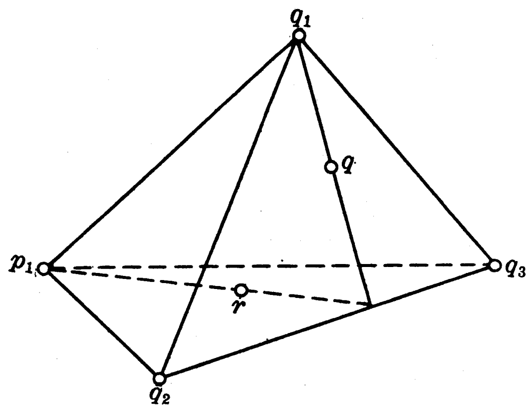

Fig. 1.

Thus the correspondence between $[p_1 + \cdots + p_n, 1]$ and the lattice of the p-fiats of the corresponding projective geometry is one-one onto. This correspondence is actually an isomorphism, because $A \subset B$ if and only if the corresponding p-flats, ϕ, φ satisfy $\phi \leq \varphi$.

If A, B are two subspaces containing $\{p, \cdots, p_n\}$ then the flat $A+B$ corresponds to the p-flat $\phi+\varphi$. Since $\phi+\varphi$ is the set union of the p-points on p-lines connecting a p-point in ϕ and another p-point in φ, one gets the following result which was also proved by Wille:

Lemma 2. Let A and B be two subspaces such that $A \cap B$ contains at least n distinct points $p_1, \cdots, p_n$. Then the lattice union $A+B$ is the set union of all surfaces $\langle p_1, \cdots, p_n, a, b\rangle$ with $a \in A$ and $b \in B$.

By the correspondence (established above) between the set of subspaces containing $p_1, \cdots, p_n$ and the p-flats of the corresponding projective geometry, one can prove these properties of the lattice of subspaces of a Wille geometry as stated in (α).

Let p be a point not in A, then $\langle p_1, \cdots, p_n, p\rangle$ defines a p-point P not in the p-flat α corresponding to A. Since $\phi+P$ covers ϕ, it follows that $A+\{p\}$ covers A. For the subspace A which contains less than n points, obviously $A+\{p\}$ covers A if $p \notin A$. Thus, the semi-modularity is proved.

The distributivity property of $[0\ a]$ for any subspace a of rank n is obvious, since each element of this lattice consists of at most n distinct points.

The modularity property of $[a, 1]$, where $a=p_1+\cdots+p_n$, follows from the fact proved above that this interval is isomorphic to the lattice of p-flats of the corresponding projective geometry.

6. Proof of (β_1) and (β_2). Call an element of rank 1 or rank 2 in the lattice $[p_1+\cdots p_n, 1]$, a point or a line. Since $\mathscr{L}$ is atomistic, upper-continuous and semi-modular, so is also the lattice $[p_1+\cdots+p_n, 1]$. That $[p_1+\cdots+p_n, 1]$ is atomistic can be shown as follows: Let a be any element in $[p_1+\cdots+p_n, 1]$. Since $\mathscr{L}$ is atomistic, $a=\Sigma S$ is a suitable set of atoms in $\mathscr{L}$. If $p \in S$, then $p_1+\cdots+p_n+p \leq a$, thus $a=\sum_{p \in S}(p_1+\cdots+p_n+p)$, where $(p_1+\cdots+p_n+p)$ is a point in $[p_1+\cdots+p_n, 1]$.

Since $[p_1+\cdots+p_n, 1]$ is modular, the set of points and lines forms a projective geometry. Now as above, if A is a subspace, the set of points $p_1+\cdots+p_n+p$ of the projective geometry which correspond to $\langle p_1, \cdots, p_n, p\rangle$ and contained in A is a flat of the projective geometry. This flat in turn corresponds to an element a in $[p_1+\cdots+p_n, 1]$ such that the flat consists of all points $p_1+\cdots+p_n+p$ with $p_1+\cdots+p_n+p \leq a$. Thus A is the set union of all curves $\langle p_1, \cdots, p_n, p\rangle$. This in turn implies that *each subspace A is the set of all atoms contained in a suitable element a of $\mathscr{L}$.*

Now $P(4)$ can be proved by using this fact: The subspace generated by $n+3$ points $p_1, \cdots, p_n, p_{n+1}, p_{n+2}, p_{n+3}$ is the set $\{\text{atom } p \mid p \leq p_1+\cdots+p_n+p_{n+1}+p_{n+2}+p_{n+3}\}$. Let A, B be two surfaces contained in the subspace $\langle p_1, \cdots, p_n, p_{n+1}, p_{n+2}, p_{n+3}\rangle$, then $A=\{\text{atom } p \mid p \leq \phi\}$, $B=\{\text{atom } p \mid p \leq \varphi\}$ with α, β elements of rank 2 in

$[p_1+\cdots+p_n, 1]$. Since $[p_1+\cdots+p_n, p_1+\cdots+p_{n+3}]$ is modular, rank $(\psi\varphi)$+rank $(\psi+\varphi)$=rank ψ+rank φ. Since rank $(\psi+\varphi)\leq 3$, so rank $(\psi\varphi)\geq 1$. That is $A\cap B$ is at least a curve, and $P(4)$ is proved.

Thus the set m of atoms of $\mathscr{L}$ together with curves and surfaces defined above is a Wille geometry. Since a subset A of m is a subspace if and only if A is the set of all atoms contained ($\leq$) in an element a of $\mathscr{L}$, by Theorem 3, the lattice $\mathscr{L}$ is isomorphic to the lattice of all subspaces of the Wille geometry obtained above.

7. It is preferable to prove the properties (α) and (β) without using the results on projective geometry (that is the special case $n=0$). For the property (α), one uses the special case of Lemma 2 to show the semi-modularity and the modularity in projective geometry. This suggests to prove the Lemma 2 directly. Actually, such a proof was given by Wille by replacing point p and line connecting p, q in the proof of projective case by curve $\langle p_1, \cdots, p_n, p\rangle$ and surface $\langle p_1, \cdots, p_n, p, q\rangle$ with n distinct fixed point $p_1, \cdots, p_n$.

For the direct proof of (β_1) and (β_2), it suffices to show directly that a subset A of m is a subspace if and only if A is the set of all atoms contained ($\leq$) in an element a of $\mathscr{L}$. As in the case of projective geometry, it suffices to show that if A is a subspace, then $A=\{\text{atom } p \mid p\leq \Sigma A\}$. For this it needs only to show that if $p\leq p_1+\cdots+p_m$, $(m\geq n+2)$, then p is contained in the subspace generated by $\{p_1, \cdots, p_m\}$. As in the special case of projective geometry, this can be shown by induction on m, counting the ranks of elements in $[p_1+\cdots+p_{n+2}, 1]$ (see [7]).

Appendix

Proof of Theorem 1. Suppose that a) is fulfilled and $X\to\bar{X}$ is the closure operation and S is closed if $\bar{K}_\gamma\subset S$ for all finite $K_\gamma\subset S$. Let M be a directed set contained in A, then every element of M is closed. One needs to show that $\cup M$ is also contained in A, that is, $\cup M$ is also closed. Let $K_\gamma\subset\cup M$ be a finite subset, so $K_\gamma=\{a_1, \cdots, a_n\}$. Since $\cup M$ is the set union there exist $m_1, \cdots, m_n$ in M such that $a_i\in m_i$ $(i=1, \cdots, n)$. Since M is directed there is an $m_0\in M$ such that $K_\gamma\subset m_0$. Since m_0 is closed, $\bar{K}_\gamma\subset m_0\subset\cup M$, hence $\cup M$ is closed.

Now assume b).

(i) A is a complete $\cap$-subband, so A is a hull-system: Since A is a complete $\cap$-subband, for any subset $B\subseteq A$, inf B formed in $P(m)$ is equal to the inf B formed in A, so it is contained in A, and A is a hull-system.

(ii) To the hull-system A, one can define a hull-operation z as follows:

$\tau: X \to \bar{X}$, where X be any element in $P(m)$ and $\bar{X}$ is the intersection of all elements of A which contain X. Then τ is easily seen to be a hull-operator.

a) If $X \leq Y$ then for each $Z \in A$ such that $Y \leq Z$, it follows that $X \leq Z$, so $\bar{X} \leq \bar{Y}$.

b) Since $\bar{X} = \pi Z$ (intersection) for all $Z \geq X$, so $\bar{X} \geq X$.

c) Since $\bar{X}$ is the intersection of all elements in A which contain X, and A is a hull-system by (i), $\bar{X} \in A$. $\bar{X}$ is in the set of elements of A which contain $\bar{X}$, so $\bar{\bar{X}} = \bar{X}$.

(iii) From the definition os $\bar{X}$, it follows that $X \in A$ implies $X = \bar{X}$. Thus, for any $K_\gamma \subseteq X$ we have $\bar{K}_\gamma \subseteq \bar{X} = X$. Let X be any element such that $\bar{K}_\gamma \subseteq X$ for all finite $K_\gamma \subseteq X$. One can show that $X \in A$, that is $x = \bar{x}$: Since $\bar{K}_\gamma \subseteq X$, so $\cup \bar{K}_\gamma \subseteq X$. Since every point of X is contained in a K_γ, hence in a $\bar{K}_\gamma$, thus $\cup \bar{K}_\gamma \supseteq X$ and hence $X = \cup \bar{K}_\gamma$. To show that $\cup \bar{K}_\gamma$ is closed, let M be the set of all $\bar{K}_\gamma$ with K_γ any finite subset of X. Then M is a subset of A (since $\bar{\bar{K}}_\gamma = \bar{K}_\gamma$), and M is a directed set, since $\bar{K}_{\gamma_1}, \bar{K}_{\gamma_2} \subseteq \overline{K_{\gamma_1} \cup K_{\gamma_2}}$ and $K_{\gamma_1} \cup K_{\gamma_2}$ is a finite subset of X. Thus $\cup \bar{K}_\gamma$ is contained in A by b). Hence $X \in A$.

It is also shown at the same time that $X \in A$ if and only if $K_\gamma \subset X$ and K_γ finite imply $\bar{K}_\gamma \subseteq X$.

Thus there is a hull-operation $X \to \bar{X}$ on $P(m)$ for which $X \in A$ if and only if $K_\gamma \subset X$ and K_γ finite imply $\bar{K}_\gamma \subset X$.

Proof of Theorem 2. Necessity (1) The lattice of subsets which are closed under a hull-operation is obviously complete,

(2) In every complete lattice, $y_\rho \leq \Sigma y_\rho$, hence $xy_\rho \leq x(\Sigma y_\rho)$ and $\Sigma xy_\rho \leq x(\Sigma y_\rho)$. Thus it remains to show that $\Sigma xy_\rho \geq x(\Sigma y_\rho)$.

By Theorem 1, for a complete $\cap$-subband V of $P(m)$, if there exists a hull-operation in $P(m)$ whose associated closure property is finitary such that V is the set of all subsets of m which are closed under the hull-operation, then for each directed set $M \subset V$, $\cup M \in V$ and hence $\Sigma M = \cup M$.

Since $\{y_\rho\}$ is a directed set of V, we have $\Sigma y_\rho = \cup y_\rho$. For $\alpha \in m$, let

$$\begin{aligned} \alpha \in x(\Sigma y_\rho) &\to \alpha \leq x \text{ and } \alpha \leq \Sigma y_\rho = U y_\rho \\ &\to \alpha \in x \text{ and } \alpha \in y_\rho \text{ for a } y_\rho \\ &\to \alpha \in x \cap y_\rho = xy_\rho \\ &\to \alpha \in \cup_M (xy_\rho) = \Sigma_M (xy_\rho)\ , \end{aligned}$$

since $\{(xy_\rho)\}$ is also a directed set contained in A, so $\cup_M (xy_\rho) = \Sigma_M (xy_\rho)$, by Theorem 1. Thus $x(\Sigma y_\rho) \leq \Sigma (xy_\rho)$.

(3) V is assumed to be isomorphic to the lattice A of all subsets of m which are closed under a hull-operation whose associated closure property is finitary. For the proof of (3), let us consider the lattice A and the set $\overline{\{\alpha\}}$ for $\alpha \in m$, where $\overline{\{\alpha\}}$ is the image of $\{\alpha\}$ under the hull-operation, so it is the smallest element of A which contains $\{\alpha\}$. For $x \in A$, $x = \Sigma\overline{\{\alpha\}}$, where $\alpha \in x$ (or $\alpha \leq x$) runs over x: For each $\alpha \leq x$, $\overline{\{\alpha\}} \leq \bar{x} = x$, so $\Sigma\overline{\{\alpha\}} \leq x$. Conversely $\alpha \in \overline{\{\alpha\}}$, so $\alpha \leq \overline{\{\alpha\}}$ and $\Sigma\alpha \leq \Sigma\overline{\{\alpha\}}$. Since $\Sigma\alpha \leq x$ and $\Sigma\alpha$ contains every point of x, $\Sigma\alpha \supseteq x$ the is $\Sigma\alpha \geq x$. Hence $x = \Sigma\alpha$ and $x = \Sigma\overline{\{\alpha\}}$. Thus (3) will be shown if one can show that each $\overline{\{\alpha\}}$ is inaccessible: Assume that $\overline{\{\alpha\}} = \Sigma y_\rho$ for a directed set $\{y_\rho\} \subset A$. By Theorem 1, $\cup y_\rho \in A$ and $\Sigma y_\rho = \cup y_\rho$. Now $\alpha \in \overline{\{\alpha\}} = \cup y_\rho$ implies that there is a y_ρ such that $\alpha \in y_\rho$, i.e., $\alpha \leq y_\rho$, hence $\overline{\{\alpha\}} \leq \bar{y}_\rho = y_\rho$. Hence $\overline{\{\alpha\}} = y_\rho$ (since $\overline{\{\alpha\}} = \cup y_\rho \supseteq y_\rho$). That is $\overline{\{\alpha\}}$ is contained in the directed set, thus $\overline{\{\alpha\}}$ is inaccessible.

Sufficiency. Conversely, suppose that a lattice V satisfies the conditions (1), (2), (3). One must show the existence of a set m and a hull-operation on $P(m)$ whose associated closure property is finitary such that V is isomorphic to the lattice A of all subsets of m which are closed under the hull-operation.

As the set m, one takes the set of all inaccessible elements u. For any subset S of m, define $\bar{S} = \{\text{inaccessible } u \mid u \leq \Sigma S\}$. Then

(i°) if $u \in S$ then $u \leq \Sigma S$ which implies $u \in \bar{S}$, hence $S \subset \bar{S}$.

(ii°) if $S_1 \subset S_2$, then $\Sigma S_1 \leq \Sigma S_2$, hence $\bar{S}_1 \subset \bar{S}_2$.

(iii°) $\bar{\bar{S}} = \bar{S}$: Since $u \in \bar{S}$ implies $u \leq \Sigma S$, so $\Sigma\bar{S} \leq \Sigma S$.

On the other hand $\bar{S} \supset S$, so $\Sigma\bar{S} \geq \Sigma S$. Hence $\Sigma\bar{S} = \Sigma S$. Since $\bar{\bar{S}} = \{\text{inaccessible } u \mid u \leq \Sigma\bar{S}\}$ and $\Sigma\bar{S} = \Sigma S$, so $\bar{\bar{S}} = \bar{S}$. Thus $S \to \bar{S}$ is a hull-operation.

It is remained to be shown that $S \to \bar{S}$ is the hull-operation whose associated closure property is finitary; that is, to show that if $K = \{u_1, \cdots, u_n\}$ is any finite subset of inaccessible elements of S then $\bar{K} \subset S$ implies that S is closed: Let y_ρ be the lattice union of finite elements of S, then the set $\{y_\rho\}$ is the directed set, and $\Sigma S = \Sigma y_\rho$, since $y_\rho \leq \Sigma S$ hence $\Sigma y_\rho \leq \Sigma S$, but on the other hand, each element of S is contained in some y_ρ, so $\Sigma S \leq \Sigma y_\rho$. Suppose that $u \in \bar{S}$, that is $u \leq \Sigma S$. Then by (2) $u = u \cdot (\Sigma S) = u \cdot (\Sigma y_\rho) = \Sigma u y_\rho$. Since $\{u y_\rho\}$ is a directed set and u is inaccessible, there is a ρ such that $u \cdot y_\rho = u$, that is, $u \leq y_\rho$, and this means that there exist $u_1, \cdots, u_n$ in S satisfying $u \leq u_1 + \cdots + u_n = y_\rho$. Thus the assumption that $y_\rho \subset S$ implies that $\bar{S} \subseteq S$, hence $S = \bar{S}$.

Let A be the set of all subsets of m which are closed under the hull-operation defined above, then A is a lattice with the three properties 1), 2), 3). Now, to each element $a \in V$, we assign the set of inaccessible elements by $\varphi(a)=\{\text{inaccessible } u \mid u \leq a\}$. To claim that this correspondence is an isomorphism between the two lattices V and A, it suffices to show that

a) φ is one-to-one,

b) $\varphi(ab)=\varphi(a) \cap \varphi(b)$,

c) for each a, $\varphi(a)$ is a closed subset of m with respect to the operation $S \rightarrow \bar{S}$,

d) to each closed subset m' of m there is an element $a \in V$ such that $\varphi(a)=m'$,

e) if $m'_1 \subseteq m'_2$ are closed, then $\varphi^{-1}(m'_1) \leq \varphi^{-1}(m'_2)$.

These are shown one by one in the following:

a) Since a is a lattice join of inaccessible elements contained in a, a is a lattice join of some elements of $\varphi(a)$, hence $a \leq \Sigma\varphi(a)$. It is shown in c) that $\Sigma\varphi(a) \leq a$. Thus $a=\Sigma\varphi(a)$. Now if $\varphi(a)=\varphi(b)$, then $a=\Sigma\varphi(a)=\Sigma\varphi(b)=b$,

b) In any lattice, $u \leq ab$ if and only if $u \leq a$ and $u \leq b$, that is if and only if $u \in \varphi(a)$ and $u \in \varphi(b)$, i.e. if and only if $u \in \varphi(a) \cap \varphi(b)$.

c) Since $\varphi(a)=\{u \mid u \leq a\}$, $\Sigma\varphi(a) \leq a$. This implies that $\overline{\varphi(a)}=\{u \mid u \leq \Sigma\varphi(a)\} \subseteq \varphi(a)$ and hence $\varphi(a)=\overline{\varphi(a)}$,

d) Let m' be any closed subset of m, and let $a=\Sigma m'$, then $m'=\bar{m}'=\{u \mid u \leq \Sigma m'=a\}=\varphi(a)$ by definition,

e) Let m'_1, m'_2 be closed subsets of m with $m'_1 \subseteq m'_2$. Let $a_1=\Sigma m'_1$ and $a_2=\Sigma m'_2$, then $m'_1=\varphi(a_1)$ and $m'_2=\varphi(a_2)$ by d). Then $\varphi^{-1}(m'_1)=a_1=\Sigma m'_1 \leq \Sigma m'_2=a_2 =\varphi^{-1}(m'_2)$, that is $\varphi^{-1}(m'_1) \leq \varphi^{-1}(m'_2)$.

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