ON THE OSCILLATION OF SOLUTIONS OF A nth-ORDER DIFFERENTIAL EQUATION

By

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1. This paper is a study of oscillation properties of the differential equation

$$y^{(n)}=p(t)y\ , \tag{L}$$

on $(-\infty, \infty)$, where $p(t)$ is positive and continuous for all t and n is an even integer greater than or equal to 4. We shall distinguish between oscillation in the positive sense and that in the negative sense. A non trivial solution of (L) is said to be positively (negatively) oscillatory if its set of zeros is not bounded above (below). It is called fully oscillatory if it is both positively oscillatory and negatively oscillatory. It is called positively (negatively) (fully) non oscillatory if it is not positively (negatively) (fully) oscillatory. Six theorems are provided, and the techniques used are similar to ones used by *Lazer* [3], *Hastings* and *Lazer* [2] and *Ahmad* [1]. Throughout this paper, we let $Z_0, Z_1, \cdots, Z_{n-1}$ denote solutions of (L) defined on $(-\infty, \infty)$ and satisfy the initial conditions

$$Z_i^{(j)}(0)=\begin{cases}0, \text{ if } i\neq j\\ 1, \text{ if } i=j\end{cases},$$

for $i, j=0, 1, \cdots, n-1$.

2. In this section, we shall give two theorems which proves the existence of two nonoscillatory solutions w and z of (L) such that $\operatorname{Sgn} w(t)=\operatorname{Sgn} w^{(j)}(t)$, $j=1, 2, \cdots, n-1$ for all $t\in(-\infty, \infty)$ and $\operatorname{Sgn} z^{(j)}(t)\neq\operatorname{Sgn} z^{(j+1)}(t)$, $j=0, 1, \cdots, n-2$, for all $t\in(-\infty, \infty)$. First, we give the following two lemmas which play important roles in obtaining the most of the results in this paper.

Lemma 2.1. If y is a non trivial solution of (L) and a is a number such that

$$y^{(j)}(a)\geq 0\ , \qquad j=0, 1, 2, \cdots, n-1\ , \tag{1}$$

then

$$y^{(j)}(t)>0\ , \qquad j=0, 1, 2, \cdots, n-1\ , \tag{2}$$

for all $t>a$.

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Proof. From (1) it is clear that the inequalities (2) hold in an interval (a, c). Let b be any number greater than a. If the inequalities (2) failed to hold in the interval $(a, b]$, then, there would be a first point c' to the right of c where $y(t)y^{(1)}(t)\cdots y^{(n-1)}(t)$ has the value zero. On the other hand

$$\begin{aligned}(y(t)y^{(1)}(t)\cdots y^{(n-1)}(t))^{(1)} = &(y^{(1)}(t))^2 y^{(2)}(t)\cdots y^{(n-1)}(t) \\ &+\cdots+y(t)y^{(1)}(t)\cdots y^{(n-3)}(t)(y^{(n-1)}(t))^2 \\ &+y(t)y^{(1)}(t)\cdots y^{(n-2)}(t)(p(t)y(t))>0\,,\end{aligned}$$

for $t\in(a, c')$.

On integrating the above inequality from a to c', we would have

$$0<\int_a^{c'}(y(t)y^{(1)}(t)\cdots y^{(n-1)}(t))^{(1)}dt=-y(a)y^{(1)}(a)\cdots y^{(n-1)}(a)\,,$$

which is a contradiction.

Lemma. 2.2. If y is a non trivial solution of (L) and a is a number such that

$$y(a)\geq 0,\ y^{(1)}(a)\leq 0,\ \cdots,\ y^{(n-1)}(a)\leq 0\,,$$

then

$$y(t)>0,\ y^{(1)}(t)<0,\ \cdots,\ y^{(n-1)}(t)<0\,,$$

for all $t<a$.

Proof. The proof of this lemma is similar to that of lemma 2.1.

We note that if y is a non trivial solution of (L), then so is $-y$. This implies from lemma 2.1 that if y is a non trivial solution of (L), and a is a number such that

$$y^{(j)}(a)\leq 0,\quad j=0, 1, \cdots, n-1\,,$$

then

$$y^{(j)}(t)<0,\quad j,=0, 1, \cdots, n-1\,,$$

for all $t>a$. Similarly from lemma 2.2, we have that if y is a non trivial solution of (L) and a is a number such that

$$y(a)\leq 0,\ y^{(1)}(a)\geq 0,\ \cdots,\ y^{(n-1)}(a)\geq 0\,,$$

then

$$y(t)<0,\ y^{(1)}(t)>0,\ \cdots,\ y^{(n-1)}(t)>0\,.$$

for all $t<a$.

Theorem 2.3. There exists a solution w of (L) such that

$$w^{(j)}(t)>0,\quad j=0, 1, \cdots, n-1\,,$$

for all $t\in(-\infty, \infty)$.

Proof. For each positive integer r, let

$$y_r = C_{0r}Z_0 + C_{1r}Z_1 + \cdots + C_{n-1r}Z_{n-1} ,$$

where

$$C_{0r}^2 + C_{1r}^2 + \cdots + C_{n-1r}^2 = 1 ,$$

$$y_r^{(j)}(-r) = C_{0r}Z_0^{(j)}(-r) + C_{1r}Z_1^{(j)}(-r) + \cdots + C_{n-1r}Z_{n-1}^{(j)}(-r) = 0 ,$$
$$j = 0, 1, \cdots, n-2 .$$

$$y_r^{(n-1)}(-r) = C_{0r}Z_0^{(n-1)}(-r) + C_{1r}Z_1^{(n-1)}(-r) + \cdots + C_{n-1r}Z_{n-1}^{(n-1)}(-r) > 0 .$$

It is easy to verify the existence of the solution y_r, which satisfies the above conditions. For, let $C'_{0r}, C'_{1r}, \cdots, C'_{n-1r}$ be numbers which satisfy the above n equations. Let

$$y_r^* = C'_{0r}Z_0 + C'_{1r}Z_1 + \cdots + C'_{n-1r}Z_{n-1} .$$

Since $Z_0, Z_1, \cdots, Z_{n-1}$ are linearly independent solutions of (L) and y_r^* is a nontrivial linear combination of these solutions, it is a nontrivial solution of (L). This implies by the uniqueness theorem that $y_r^{*(n-1)}(-r) \neq 0$. Now if $y_r^{*(n-1)}(-r) > 0$, then we can take $y_r(t) = y_r^*(t)$. If $y_r^{*(n-1)}(-r) < 0$, then we can take $y_r(t) = -y_r^*(t)$.

Since the sequences $\langle C_{ir} \rangle$, $i = 0, 1, \cdots, n-1$, are bounded, there exists a sequence $\langle r_k \rangle$ of positive integers such that the subsequences $\langle C_{ir_k} \rangle$, $i = 0, 1, \cdots, n-1$, converge to the numbers C_i, $i = 0, 1, \cdots, n-1$ which satisfy

$$C_0^2 + C_1^2 + \cdots + C_{n-1}^2 = 1 .$$

We now consider the solution

$$w = C_0Z_0 + C_1Z_1 + \cdots + C_{n-1}Z_{n-1} .$$

If $w^{(j)}(t_0) < 0$ for some $j = 0, 1, \cdots, n-1$ and for some number t_0, then since $\langle y_{r_k}^{(j)}(t_0) \rangle$ converges to $w^{(j)}(t_0)$, there exists a positive N such that

$$y_{r_k}^{(j)}(t_0) < 0 \text{ for all } r_k > N .$$

But this leads to a contradiction, since for $-r_k < t_0$,

$$y_{r_k}^{(j)}(-r_k) \geq 0 \text{ for all } j = 0, 1, \cdots, n-1 ,$$

and by lemma 2.1 $y_{r_k}^{(j)}(t_0) > 0$. Consequently

$$w^{(j)}(t) \geq 0, \text{ for all } j = 0, 1, \cdots, n-1 \text{ and for all } t.$$

Now, since w is a nontrivial solution of (L), there is no number τ such that $w^{(j)}(\tau) = 0$ for all $j = 0, 1, \cdots, n-1$. Hence again lemma 2.1 implies that

$$w^{(j)}(t) > 0, \text{ for all } j = 0, 1, \cdots, n-1 \text{ and for all } t.$$

Theorem 2.4. There exists a solution z of (L) such that

$$z(t)>0,\ z^{(1)}(t)<0, \cdots, z^{(n-1)}(t)<0\ ,$$

for all $t\in(-\infty, \infty)$.

Proof. The proof of this theorem is similar to that of theorem 2.3. We modify each y_r to satisfy the conditions

$$y_r^{(j)}(r)=0\ , \qquad j=0, 1, \cdots, n-2\ ,$$

$$y_r^{(n-1)}(r)<0\ ,$$

and use lemma 2.2.

3. Positive and negative oscillations.

Here we shall consider positively oscillatory and negatively oscillatory solutions of (L). We observe that $y(t)$ is a negatively oscillatory solution of (L) if and only if $Y(t)$ is a positively oscillatory solution of

$$Y^{(n)}=P(t)Y\ ,$$

where $Y(t)=y(-t)$ and $P(t)=p(-t)$. Further,

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $t\in(-\infty, t_0]$ (and hence for all t, by lemma 2.1) if and only if

$$\operatorname{Sgn} Y^{(j)}(t)\neq\operatorname{Sgn} Y^{(j+1)}(t)\ , \qquad j=0, 1, \cdots, n-2\ ,$$

for all $t\in[-t_0, \infty)$ (and hence for all t, by lemma 2.2). Similary,

$$\operatorname{Sgn} y^{(j)}(t)\neq\operatorname{Sgn} y^{(j+1)}(t)\ , \qquad j=0, 1, 2, \cdots, n-2\ ,$$

for all $t\in(-\infty, t_0]$ if and only if

$$\operatorname{Sgn} Y(t)=\operatorname{Sgn} Y^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $t\in[-t_0, \infty)$.

Thus the study of negatively oscillatory solutions of (L), can be reduced to that of positively oscillatory solutions. We shall state the results about negatively oscillatory solutions as corollaries whose proofs follow from the above observations and the corresponding results of positively oscillatory solutions.

Our first theorem of this section proves the existence of $n-1$ positively oscillatory solutions of (L).

Theorem 3.1. If for every positively non oscillatory solution y of (L) either there exists a number t_0 such that

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $t \geq t_0$, or

$$\operatorname{Sgn} y^{(j)}(t) \neq \operatorname{Sgn} y^{(j+1)}(t)\ , \quad j=0, 1, 2, \cdots, n-2\ .$$

for all $t \in (-\infty, \infty)$, then there exists $n-1$ linearly independent positively oscillatory solutions of (L).

Proof. For each positive integer r, let μ_{ir}, $i=0, 1, \cdots, n-2$ be the solutions of (L) given by

$$\mu_{ir}=a_{ir}Z_i+b_{ir}Z_{n-1}\ ,$$

where

$$a_{ir}^2+b_{ir}^2=1\ ,$$

and

$$a_{ir}Z_i(r)+b_{ir}Z_{n-1}(r)=0\ .$$

Now there exists a sequence $\langle r_k \rangle$ of positive integers such that the subsequences $\langle a_{ir_k} \rangle$ and $\langle b_{ir_k} \rangle$ converge to numbers a_i and b_i respectively, which satisfy

$$a_i^2+b_i^2=1\ .$$

We now consider the solutions μ_i, $i=0, 1, \cdots, n-2$ given by

$$\mu_i=a_iZ_i+b_iZ_{n-1}\ ,$$

Suppose μ_i is positively non oscillatory. Since for a positive integer $k<n-1$ and $k \neq i$, $\mu^{(k)}(0)=0$, we cannot have $\mu_i^{(j)}(t) \neq \mu_i^{(j+1)}(t)$, $j=0, 1, \cdots, n-2$, for all $t \in (-\infty, \infty)$. Consequently, there exists a number t_0, such that

$$\operatorname{Sgn} \mu_i(t)=\operatorname{Sgn} \mu_i^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ , \quad \text{for all } t>t_0,$$

Now let τ be any number greater than t_0. Since $\langle \mu_{ir_k}^{(j)}(\tau) \rangle$ converges to $\mu_i^{(j)}(\tau)$, $j=0, 1, 2, \cdots, n-1$, there exists a positive integer N such that

$$\operatorname{Sgn} \mu_{ir_k}(\tau)=\operatorname{Sgn} \mu_{ir_k}^{(j)}(\tau)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $r_k>N$. But this is a contradiction, since $\mu_{ir_k}(r_k)=0$ for positive integer r_k. Therefore, the solutions $\mu_i=a_iZ_i+b_iZ_{n-1}$, $i=0, 1, \cdots, n-2$, are positively oscillatory.

Now to prove that the $n-1$ positively oscillatory solutions $\mu_0, \mu_1, \cdots, \mu_{n-1}$ are linearly independent, it is sufficient to show that $a_i \neq 0$, $i=0, 1, \cdots, n-2$. For if some $a_i=0$, then

$$\mu_i=b_iZ_{n-1}\ .$$

Consequently

$$\mu_i^{(j)}(0)=0\,, \qquad j=0,1,\cdots,n-2\,,$$

$$\mu_i^{(n-1)}(0)=b_i\,.$$

This implies by lemma 2.1 that there exists t_0 such that for all $t>t_0$, $\mu_i^{(j)}(t)>0$ or $\mu_i^{(j)}(t)<0$, $j=0,1,\cdots,n-1$. Thus μ_i is not positively oscillatory, which is a contradiction. Hence

$$a_i\neq 0\,, \qquad i=0,1,\cdots,n-2\,.$$

Corollary 3.2. If for every negatively non oscillatory solution y of (L) either there exists a number t_0 such that

$$\operatorname{Sgn} y^{(j)}(t)\neq \operatorname{Sgn} y^{(j+1)}(t)\,, \qquad j=0,1,\cdots,n-2\,,$$

for all $t\in(-\infty, t_0]$ or

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\,, \qquad j=1,2,\cdots,n-1\,,$$

for all $t\in(-\infty,\infty)$, then there exists $n-1$ linearly independent negatively oscillatory solutions of (L).

The following theorem gives a necessary condition for the uniqueness of the solution z given in theorem 2.4.

Theorem 3.3. Let (L) has no positively oscillatory solution. If y is any solution of (L) such that

$$\operatorname{Sgn} y^{(j)}(t)\neq \operatorname{Sgn} y^{(j+1)}(t)\,, \qquad j=0,1,2,\cdots,n-2\,,$$

for all t, then $y=cz$ where z is the solution of (L) given in Theorem 2.4 and c is a number.

Proof. If y and z are linearly dependent, then the theorem is trivial. Suppose that y and z are linearly independent. Let $\upsilon=z+cy$, where c is a number such that $\upsilon(0)=z(0)+cy(0)=0$. Since υ is positively non-oscillatory, there exists a number t_0 such that none of $\upsilon, \upsilon^{(1)},\cdots,\upsilon^{(n-1)}$ change sign on $[t_0,\infty)$. Assume, without loss of generality that υ and hence $\upsilon^{(n)}$ are positive on $[t_0,\infty)$. Since y and z are both bounded, υ is bounded and thus we must have $\upsilon^{(n-1)}(t)<0$, $\upsilon^{(n-2)}(t)>0,\cdots,\upsilon^{(1)}(t)<0$, for all $t\geq t_0$. Consequently by lemma 2.2, $\upsilon(t)>0$, $\upsilon^{(1)}(t)<0,\cdots,\upsilon^{(n-1)}(t)<0$ for all $t\in(-\infty,\infty)$. But this is a contradiction, since $\upsilon(0)=0$. This proves the theorem.

Corollary 3.4. Let (L) has no negatively oscillatory solution. If y is any solution of (L) such that

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\,, \qquad j=0,1,\cdots,n-2\,,$$

for all $t\in(-\infty, \infty)$, then $y=cw$, where w is the solution of (L) given in Theorem 2.3 and c is number.

Theorem 3.5. Let $y_1, y_2, \cdots, y_{n-1}$ be any $(n-1)$ linearly independent solutions (L). If every positively non oscillatory solution y of (L) is such that either there exists a number t_0 such that

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\,, \qquad j=1, 2, \cdots, n-1\,,$$

for all $t\geq t_0$, or

$$\operatorname{Sgn} y^{(j)}(t)\neq\operatorname{Sgn} y^{(j+1)}(t)\,, \quad j=0, 1, 2, \cdots, n-2\,,$$

for all $t\in(-\infty, \infty)$, then some linear combination of $y_1, y_2, \cdots, y_{n-1}$ is positively non oscillatory.

Proof. We assert that $w, z, \mu_0, \mu_1, \cdots, \mu_{n-3}$ from a basis for the solutions of (L), where $\mu_0, \mu_1, \cdots, \mu_{n-3}$ are the positively oscillatory solutions of theorem 3.1 and w and z are solutions of theorem 2.3 and theorem 2.4 respectively. For, z is bounded on $[0, \infty)$ and $\lim_{t\to\infty} w(t)=\infty$, while $w(t)$ is bounded on $(-\infty, 0]$ and $\lim_{t\to-\infty} z(t)=\infty$. Thus we can write $y_1, y_2, \cdots, y_{n-1}$ as

$$y_i=a_{i0}\mu_0+a_{i1}\mu_1+\cdots+a_{in-3}\mu_{n-3}+a_{in-2}w+a_{in-1}z\,, \qquad i=1, 2, \cdots, n-1\,.$$

Let $d_1, d_2, \cdots, d_{n-1}$ be numbers (not all zeros) such that

$$d_1a_{10}+d_2a_{20}+\cdots+d_{n-1}a_{n-10}=0$$
$$d_1a_{11}+d_2a_{21}+\cdots+d_{n-1}a_{n-11}=0$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots$$
$$d_1a_{1n-3}+d_2a_{2n-3}+\cdots d_{n-1}a_{n-1,n-3}=0\,.$$

Then $y=d_1y_1+d_2y_2+\cdots d_{n-1}y_{n-1}$ is a linear combination of w and z and hence positively non oscillatory.

Corollary 3.6. Let $y_1, y_2, \cdots, y_{n-1}$ be any $(n-1)$ linearly independent solutions of (L). If every negatively non oscillatory solution y of (L) is such that

$$\operatorname{Sgn} y^{(j)}(t)\neq\operatorname{Sgn} y^{(j+1)}(t)\,, \qquad j=0, 1, 2, \cdots, n-2\,,$$

for all $t\in(-\infty, t_0)$, or

$$\operatorname{Syn} y(t)=\operatorname{Sgn} y^{(j)}(t)\,, \qquad j=1, 2, \cdots, n-1,$$

for all $t\in(-\infty, \infty)$, then some linear combination of $y_1, y_2, \cdots, y_{n-1}$ is negatively non oscillatory.

4. Full Oscillation.

The following theorem proves the existence of $n-2$ linearly independent fully oscillatory solutions of (L).

Theorem 4.1. If for every positively non oscillatory solution of (L) either there exists a number t_0 such that

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $t \geq t_0$, or

$$\operatorname{Sgn} y^{(j)}(t) \neq \operatorname{Sgn} y^{(j+1)}(t)\ , \qquad j=0, 1, \cdots, n-2\ ,$$

for all $t \in (-\infty, \infty)$, and if for every negatively non oscillatory solution y of (L) either there exists a number t_0 such that

$$\operatorname{Sgn} y^{(j)}(t) \neq \operatorname{Sgn} y^{(j+1)}(t)\ , \qquad j=0, 1, \cdots, n-2\ ,$$

for all $t \in (-\infty, t_0)$ or

$$\operatorname{Sgn} y(t)=\operatorname{Sgn} y^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ ,$$

for all $t \in (-\infty, \infty)$, then it has $n-2$ linearly independent fully oscillatory solutions.

Proof. For each positive integer, r, let U_{ir}, $i=0, 1, \cdots, n-3$ be the solutions of (L) defined by

$$U_{ir}=a_{ir}Z_i+b_{ir}Z_{n-2}+c_{ir}Z_{n-1}\ ,$$

where the numbers a_{ir}, b_{ir}, and c_{ir} satisfy the following equations.

$$a_{ir}^2+b_{ir}^2+c_{ir}^2=1\ ,$$

$$a_{ir}Z_i(r)+b_{ir}Z_{n-2}(r)+c_{ir}Z_{n-1}(r)=0\ ,$$

$$a_{ir}Z_i(-r)+b_{ir}Z_{n-2}(-r)+c_{ir}Z_{n-1}(-r)=0\ .$$

Since the sequences $\langle a_{ir}\rangle$, $\langle b_{ir}\rangle$ and $\langle c_{ir}\rangle$ are bounded, there exists a sequence $\langle r_k\rangle$ of positive integers such that the subsequences $\langle a_{ir_k}\rangle$, $\langle b_{ir_k}\rangle$ and $\langle c_{ir_k}\rangle$ converge to numbers a_i, b_i and c_i respectively, which satisfy

$$a_i^2+b_i^2+c_i^2=1\ , \qquad i=0, 1, 2, \cdots, n-3\ .$$

We now consider the solution U_i, $i=0, 1, 2, \cdots, n-3$ defined by

$$U_i=a_iZ_i+b_iZ_{n-2}+c_iZ_{n-1}\ .$$

One can verify, using an argument similar to ones used in proofs of theorem 3.1 and corollary 3.2 that U_i, $i=0, 1, \cdots, n-3$ are fully oscillatory.

Now to prove that the $n-2$ fully oscillatory solutions $U_0, U_1, \cdots, U_{n-3}$ are linearly independent, it is sufficient to show that $a_i \neq 0$, $i=0, 1, \cdots, n-3$. For if some $a_i=0$, then $U_i(t)=b_i Z_{n-2}(t)+c_i Z_{n-1}(t)$.

This implies

$$U_i^{(j)}(0)=0\ , \qquad j=0, 1, 2, \cdots, n-3\ ,$$

$$U_i^{(n-2)}(0)=b_i\ ,$$

$$U_i^{(n-1)}(0)=c_1\ .$$

If $\operatorname{Sgn} b_i = \operatorname{Sgn} c_i$, then by lemma 2.1, there exists t_0 such that for all $t>t_0$.

$$\operatorname{Sgn} U_i(t)=\operatorname{Sgn} U_i^{(j)}(t)\ , \qquad j=1, 2, \cdots, n-1\ .$$

Again, if $\operatorname{Sgn} b_i \neq \operatorname{Sgn} c_i$, then lemma 2.2, for all $t<0$,

$$\operatorname{Sgn} U_i^{(j)}(t) \neq \operatorname{Sgn} U_i^{(j+1)}(t)\ .$$

Thus U_i is not fully oscillatory solution of (L), which is a contradiction.

REFERENCES

[1] Shair Ahmad, On the oscillations of solutions of a class of linear fourth order differential equations, Pacific J. Math. 34 (1970), 289-299.

[2] S.P. Hastings, and A.C. Lazer, On the asymptotic behavior of solutions of the differential equation $y^{(4)}=p(t)y$, Czech. Math. J. 93 (1968), 224-229.

[3] A.C. Lazer, The behavior of solutions of the differential equation $y'''+p(x)y'+q(x)y=0$, Pacific J. Math. 17 (1966), 435-466.

[4] C.A. Swanson, Comparison and oscillation theory of linear differential equations, Academic Press, New York, (1968).

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