CURVATURE TENSORS AND THEIR RELATIVISTIC SIGNIFICANCE III

By

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Summary:—In this paper new curvature tensors have been defined and their various physical and geometrical properties are studied.

1. Introduction. In the n-dimensional space V_n, the tensors.

$$C(X, Y, Z, T)=R(X, Y, Z, T)-\frac{R}{n(n-1)}[g(X, T)g(Y, Z)-g(Y, T)g(X, Z)], \tag{1.1}$$

$$\begin{aligned}L(X, Y, Z, T)=R(X, Y, Z, T)-\frac{1}{n-2}[&g(Y, Z)\operatorname{Ric}(X, T)\\ &-g(X, Z)\operatorname{Ric}(Y, T)+g(X, T)\operatorname{Ric}(Y, Z)-g(Y, T)\operatorname{Ric}(X, Z)],\end{aligned} \tag{1.2}$$

and

$$\begin{aligned}V(X, Y, Z, T) = R(X, Y, Z, T) - \frac{1}{n-2}[&g(Y, Z)\operatorname{Ric}(X, T)\\ &-g(X, Z)\operatorname{Ric}(Y, T)+g(X, T)\operatorname{Ric}(Y, Z)-g(Y, T)\operatorname{Ric}(X, Z)]\\ &+\frac{R}{(n-1)(n-2)}[g(X, T)g(Y, Z)-g(Y, T)g(X, Z)],\end{aligned} \tag{1.3}$$

are called concircular curvature tensor, conharmonic curvature tensor and conformal curvature tensor respectively [1]. These satisfy the symmetric and skew symmetric as well as the cyclic property possessed by the curvature tensor $R(X, Y, Z, T)$.

The projective curvature tensor is given by

$$\begin{aligned}W(X, Y, Z, T)=R(X, Y, Z, T)+\frac{1}{n-1}\\ \times[g(X, Z)\operatorname{Ric}(Y, T)-g(X, T)\operatorname{Ric}(Y, Z)].\end{aligned} \tag{1.4}$$

In our recent papers [1], [2], we have defined some curvature tensors and explored their various physical and geometrical properties.

Here we shall define two new tensors and obtain their properties.

2. Definition. We define the tensors

$$W_3(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) + \frac{1}{n-1} \times [g(Y, Z)\,\mathrm{Ric}\,(X, T) - g(Y, T)\,\mathrm{Ric}\,(X, Z)]\,, \tag{2.1}$$

and

$$W_4(X, Y, Z, T) \overset{\text{def}}{=} R(X, Y, Z, T) + \frac{1}{n-1} \times [g(X, Z)\,\mathrm{Ric}\,(Y, T) - g(X, Y)\,\mathrm{Ric}\,(Z, T)]\,. \tag{2.2}$$

From equations (1.1) to (2.2), it is clear that for an empty gravitational field characterized by $\mathrm{Ric}\,(X, Y)=0$, the six fourth rank tensors are identical.

We notice from (2.1) that $W_3(X, Y, Z, T)$ is skew-symmetric in Z, T and

$$W_3(X, Y, Z, T) + W_3(Y, Z, X, T) + W_3(Z, X, Y, T) \neq 0\,. \tag{2.3}$$

Breaking $W_3(X, Y, Z, T)$ into two parts

$$\alpha(X, Y, Z, T) = \frac{1}{2}[W_3(X, Y, Z, T) - W_3(Y, X, Z, T)]\,,$$

and

$$\beta(X, Y, Z, T) = \frac{1}{2}[W_3(X, Y, Z, T) + W_3(Y, X, Z, T)]\,,$$

which are respectively skew-symmetric and symmetric in X, Y.

From (2.1), it follows that

$$\alpha(X, Y, Z, T) = R(X, Y, Z, T) + \frac{1}{2(n-1)}[g(Y, Z)\,\mathrm{Ric}\,(X, T) - g(Y, T)\,\mathrm{Ric}\,(X, Z) - g(X, Z)\,\mathrm{Ric}\,(Y, T) + g(X, T)\,\mathrm{Ric}\,(Y, Z)]\,, \tag{2.4}$$

and

$$\beta(X, Y, Z, T) = \frac{1}{2(n-1)}[g(Y, Z)\,\mathrm{Ric}\,(X, T) - g(Y, T)\,\mathrm{Ric}\,(X, Z) + g(X, Z)\,\mathrm{Ric}\,(Y, T) - g(X, T)\,\mathrm{Ric}\,(Y, Z)]\,. \tag{2.5}$$

From (2.4), we see that $\alpha(X, Y, Z, T)$ possesses all the symmetric and skew symmetric properties of $R(X, Y, Z, T)$ as well as the cyclic property

$$\alpha(X, Y, Z, T) + \alpha(Y, Z, X, T) + \alpha(Z, X, Y, T) = 0\,. \tag{2.6}$$

From equations (1.3) and (2.5), we get

(2.7) $$\alpha(X, Y, Z, T)=R(X, Y, Z, T)+\frac{1}{2(n-1)}\times\Big[(n-2)\{R(X, Y, Z, T)-V(X, Y, Z, T)\}+\frac{R}{(n-1)(n-2)}\{g(X, T)g(Y, Z)-g(X, Z)g(Y, T)\}\Big],$$

which for electromagnetic field (or more generally in the case of space with vanishing scalar curvature) in V_4 becomes

(2.8) $$3\alpha(X, Y, Z, T)=4R(X, Y, Z, T)-V(X, Y, Z, T),$$

also from equations (1.2) and (2.5), for V_4, we have

(2.9) $$3\alpha(X, Y, Z, T)=4R(X, Y, Z, T)-L(X, Y, Z, T).$$

Thus equation (2.8) is the consequence of (2.9) for an electromagnetic field.

We notice that the symmetric part $\beta(X, Y, Z, T)$ is identically equal to the symmetric part of Weyl projective curvature tensor [3] where as its skew-symmetric part is different from $\alpha(X, Y, Z, T)$.

From equations (1.2), (1.4) and (2.1) we get

(2.9)b $$W_8(X, Y, Z, T)=W(X, Y, Z, T)+\frac{n-2}{n-1}[R(X, Y, Z, T)-L(X, Y, Z, T)],$$

which for V_4 becomes

(2.9)c $$W_8(X, Y, Z, T)=W(X, Y, Z, T)+\frac{2}{3}[R(X, Y, Z, T)-L(X, Y, Z, T)].$$

On contracting W_{8hijk} defined by (2.1) we get

(2.10) $$W_{8ij}=\left(\frac{n-2}{n-1}\right)\left(R_{ij}+\frac{R}{n-2}g_{ij}\right),$$

and the scalear invariant

(2.11) $$W_8\equiv g^{ij}W_{8ij}=2R.$$

The scalar invariant of second degree in W_{8ij} is given by

(2.12) $$(W_8)_{II}=W_{8ij}W_8^{ij}=\left(\frac{n-2}{n-1}\right)^2\left(R_2+\frac{2R^2}{n-2}+\frac{nR^2}{(n-2)^2}\right),$$

where

$$R_2=R_{ij}R^{ij}.$$

From (2.4), on contracting, we get

$$\alpha_{ij}=\frac{3n-4}{2(n-1)}\left(R_{ij}+\frac{R}{3n-4}g_{ij}\right), \tag{2.13}$$

and

$$\alpha=g^{ij}\alpha_{ij}=2R\,. \tag{2.14}$$

From (2.13) we notice that α_{ij} does not vanish in an Einstein space. Thus it is not possible to extend the Pirani formalism of gravitational wave to the Einstein space with the help of α_{hijk}.

Similar from (2.5), on contraction, we get

$$\beta_{hk}=\frac{n}{2(n-1)}\left(R_{hk}-\frac{R}{n}g_{hk}\right), \tag{2.15}$$

and the scalar invariant β defined by

$$\beta=g^{hk}\beta_{hk}\,,$$

vanishes identically in view of (2.15).

If we substitute for R_{hk} from (2.15) into (2.5), we get

$$\beta_{hijk}=\frac{1}{n}(g_{hj}\beta_{ik}-g_{hk}\beta_{ij}+g_{ij}\beta_{hk}-g_{ik}\beta_{hj})\,. \tag{2.16}$$

Thus the vanishing of β_{hijk} is the necessary and sufficient condition for a space to be an Einstein space.

From (2.2), we notice that $W_4(X, Y, Z, T)$ has no symmetry, but it satisfies cyclic property

$$W_4(X, Y, Z, T)+W_4(Y, Z, X, T)+W_4(Z, X, Y, T)=0\,. \tag{2.17}$$

On contraction, it reduces to Ricci tensor *i.e.*

$$W_{4ij}=R_{ij}\,. \tag{2.18}$$

The vector

$$\theta_i=\frac{g_{ij}\epsilon^{jklm}R_k{}^pR_{pl:m}}{\sqrt{-g}R_{ab}R^{ab}}, \tag{2.19}$$

is called the complexion vector of a non-null electromagnetic field with no matter by *Misner and Wheeler* [4] and its vanishing implies that field is purely electrical. A semicolon stands for convariant differentiation.

Interchanging the dummy indices l, m (2.19) can be written as

$$\begin{aligned}\theta_i&=\frac{g_{ij}\epsilon^{jkml}R_k{}^pR_{pm;l}}{\sqrt{-g}R_{ab}R_o{}^b},\\ &=-\frac{g_{ij}\epsilon^{jklm}R_k{}^pR_{pm;l}}{\sqrt{-g}R_{ab}R^{ab}}\,.\end{aligned} \tag{2.20}$$

By setting $W^{h}_{4pml;h}=0$, we get

$$R_{pm;l}=R_{pl;m} , \tag{2.21}$$

which on substitution in (2.20) implies that $\theta_i=0$. Thus the vanishing of the divergence of W^{h}_{4ijk} in an electromagnetic field implies a purely electric field.

It is seen that we cannot get a purely electric field with the help of W_{8hijk}.

Rainich [5] has shown that the necessary and sufficient condition for the existence of the non-null electrovariance are

$$R=0 , \tag{2.22}$$

$$R_j{}^iR_k{}^j=\frac{1}{4}\delta_k{}^iR_{ab}R^{ab} , \tag{2.23}$$

$$\theta_{i;j}=\theta_{j;i} . \tag{2.24}$$

From (2.18) we notice that W_{4ij} can very well be substituted in place of R_{ij} in the above conditions.

For an electromagnetic field, from (2.10), we get

$$W_{8ij}=\frac{2}{3}R_{ij} . \tag{2.25}$$

Thus W_{8ij} can also replace R_{ij} in the Rainich conditions.

Thus from the above discussion, we conclude that the new defined tensors can very well be used in place of *Weyl* projective tensor in various physical and geometrical spheres.

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