ON THE LOGARITHMIC CONVERGENCE EXPONENT OF THE ZEROS OF ENTIRE FUNCTIONS

By

P.K. JAIN* and V.D. CHUGH

(Received March 8, 1972)

1. Introduction. If $f(z)$ is an entire function of order ρ with an m-fold zero at the origin, then by Hadamard's Factorization Theorem we have

(1.1) $$f(z)=z^m\cdot e^{Q(z)}\cdot P(z) ,$$

where $P(z)$ is the canonical product (of genus p) formed with the zeros (other than $z=0$) of $f(z)$; $Q(z)$ is a polynomial of degree $q\leqslant\rho$. The order of $P(z)$ is $\rho_1\leqslant\rho$, ρ_1 being the convergence exponent of the zeros of $f(z)$. The genus of $f(z)$ is the max (p, q), (see [1], Ch. II). In particular, if $l=0$, then $q=0$, $p=0$, $\rho_1=0$ and the genus of $f(z)$ is also zero.

Throughout this paper we shall assume $f(z)$ to be a nonconstant entire function of order zero. For an entire function $f(z)$ of this nature, the logarithmic order ρ^* and the lower logarithmic order λ^* are given as (see [2]):

$$\lim_{r\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(r, f)}{\log\log r} = \begin{matrix}\rho^*\\ \lambda^*\end{matrix} \quad (1\leqslant\lambda^*\leqslant\rho^*\leqslant\infty) ,$$

where $M(r,f)=\max_{|z|=r} |f(z)|$. It is worth noting that for all entire functions $\lambda^*\geqslant 1$.

Further, if $\{r_n\}_{n=1}^{\infty}$ denotes the sequence of the moduli of the zeros of $f(z)$, then

$$\rho_1=\text{g.l.b.}\{\alpha:\ \alpha>0 \text{ and } \sum_{n=1}^{\infty} r_n^{-\alpha}<\infty\}=0 .$$

To have a more precise description of the distribution of the zeros of such functions, let us consider

$$\rho_1^*=\text{g.l.b. } \{\alpha:\ \alpha>0 \text{ and } \sum_{n=1}^{\infty} (\log r_n)^{-\alpha}<\infty\} .$$

In analogy with the convergence exponent, ρ_1, of the zeros of $f(z)$, ρ_1^* will be

* The work of this author has been supported partially by the University Grants Commission (India).

called the logarithmic convergence exponent of the zeros of $f(z)$.

Our aim in this paper is to study the growth relations of entire functions of logarithmic order with respect to the distribution of its zeros. The most striking result of the paper is that the canonical product $P(z)$ is of logarithmic order ρ_1^*+1 which further equals to ρ^*.

2. Theorem. *If $f(z)$ has at least one zero, then*

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log\log r}=\rho_1^* ,$$

where $n(r)$ is the number of the zeros of $f(z)$ in $|z|\leqslant r$.*

To prove this we need the following lemma:

Lemma 1. *The series*

$$\sum_{n=1}^{\infty} (\log r_n)^{-\alpha} \tag{2.1}$$

and the integral

$$\int_1^{\infty} \frac{n(x)}{(\log x)^{\alpha+1}} \frac{dx}{x} , \tag{2.2}$$

converge or diverge together if $\alpha>0$.

Proof. A partial sum of the series (2.1) is

$$\int_1^R \frac{dn(x)}{(\log x)^{\alpha}}=\frac{n(R)}{(\log R)^{\alpha}}+\alpha\int_1^R \frac{n(x)}{(\log x)^{\alpha+1}} \frac{dx}{x} . \tag{2.3}$$

If the left-hand side is bounded as $R\to\infty$, the integral on the right-hand side does not exceed that on the left and hence (2.2) converges.

Further, if (2.2) converges, we have

$$\int_1^{R^2} \frac{n(x)}{(\log x)^{\alpha+1}} \frac{dx}{x}>\int_R^{R^2} \frac{n(x)}{(\log x)^{\alpha+1}} \frac{dx}{x}>n(R)\int_R^{R^2} \frac{1}{(\log x)^{\alpha+1}} \frac{dx}{x}$$
$$=\alpha^{-1}\cdot n(R)(1-2^{-\alpha})(\log R)^{-\alpha} ,$$

which implies

$$n(R)=O((\log R)^{\alpha}) .$$

Thus the right-hand side of (2.3) is bounded and therefore so is the left-hand side, *i.e.* (2.1) converges.

Proof of the theorem. Let

---

* We assume, without any loss of generality, that $n(r)=0$ for $r\leqslant 1$.

$$\limsup_{r\to\infty} \frac{\log n(r)}{\log\log r} = \delta .$$

Therefore

$$n(r) = O((\log r)^{\delta+\varepsilon}), \quad r \geqslant r_0(\varepsilon), \quad \varepsilon > 0;$$

so (2.2) and hence (2.1), in view of the above lemma, is convergent if $\alpha > \delta + \varepsilon$. This means that $\rho_1^* \leqslant \delta$.

Also, $\exists$ an increasing sequence $\{r_k\}_{k=1}^{\infty}$ such that

$$n(r_k) > (\log r_k)^{\delta-\varepsilon} .$$

Therefore, for $R > (r_k)^{2^{1/\alpha}}$, we have

$$\begin{aligned}\int_{r_k}^{R} \frac{n(x)}{(\log x)^{\alpha+1}} \frac{dx}{x} &\geqslant n(r_k) \int_{r_k}^{R} \frac{1}{(\log x)^{\alpha+1}} \frac{dx}{x} \\ &> (\log r_k)^{\delta-\varepsilon} \int_{r_k}^{R} \frac{1}{(\log x)^{\alpha+1}} \frac{dx}{x} \\ &= \alpha^{-1} \cdot (\log r_k)^{\delta-\varepsilon}((\log r_k)^{-\alpha} - (\log R)^{-\alpha}) \\ &> (2\alpha)^{-1}(\log r_k)^{\delta-\alpha-\varepsilon}\end{aligned}$$

which implies that the integral (2.2) and hence the series (2.1) is divergent for $\alpha < \delta - \varepsilon$. Hence $\rho_1^* \geqslant \delta$.

This proves the theorem.

Theorem 2. *A canonical $P(z)$ is an entire function of logarithmic order ρ_1^*+1.*

For this we first prove:

Lemma 2. *If*

$$N(r) = \int_0^r \frac{n(t)}{t} dt ,$$

then

$$\limsup_{r\to\infty} \frac{\log N(r)}{\log\log r} = \rho_1^* + 1 .$$

Lemma 3. *For every entire function of logarithmic order ρ^* and logarithmic convergence exponent ρ_1^*, $\rho^* \geqslant \rho_1^*+1$.*

Proof of the lemmas. The lemma 2 follows from theorem 1 and the inequalities

$$N(r^2) \geqslant \int_r^{r^2} \frac{n(t)}{t} dt \geqslant n(r) \int_r^{r^2} \frac{dt}{t} = n(r) \cdot \log r ,$$

and

$$N(r) \leqslant n(r) \cdot \log r(1+o(1)) .$$

Further, by *Jensen's theorem*, (see [1], (2.5.9), p. 15), we have

$$N(r) \leqslant \log M(r, f).$$

Hence lemma 3 follows from the above inequality by using lemma 2.

Proof of the theorem. It is already known that $P(z)$ is always an entire function (see [1], p. 19). Let σ be the logarithmic order of $P(z)$. Therefore, by lemma 2, $\sigma \geqslant \rho_1^*+1$. To prove the reverse inequality, write (see [1], p. 19)

$$\log |P(z)| \leqslant \int_0^r \frac{n(t)}{t}\, dt + r \int_r^\infty \frac{n(t)}{t^2}\, dt .$$

Also by Theorem 1,

$$n(t) \leqslant (\log t)^{\rho_1^*+\varepsilon}, \quad r \geqslant r_0(\varepsilon), \quad \varepsilon > 0 .$$

Therefore

$$\log |P(z)| \leqslant \frac{(\log r)^{\rho_1^*+1+\varepsilon}}{\rho_1^*+1+\varepsilon} + r \int_r^\infty (\log t)^{\rho_1^*+\varepsilon} \frac{dt}{t^2} + O(1) .$$

Now, consider

$$\begin{aligned} I &= \int_r^\infty (\log t)^{\rho_1^*+\varepsilon} \frac{dt}{t^2} \\ &= \frac{(\log r)^{\rho_1^*+\varepsilon}}{r} + (\rho_1^* + \varepsilon) \int_r^\infty (\log t)^{\rho_1^*-1+\varepsilon} \frac{dt}{t^2} \\ &\leqslant \frac{(\log r)^{\rho_1^*+\varepsilon}}{r} + \frac{(\rho_1^*+\varepsilon)}{\log r} \int_r^\infty (\log t)^{\rho_1^*+\varepsilon} \frac{dt}{t^2} \\ &= \frac{(\log r)^{\rho_1^*+\varepsilon}}{r} + \frac{(\rho_1^*+\varepsilon)}{\log r} I \end{aligned}$$

which implies

$$r \cdot I \leqslant (\log r)^{\rho_1^*+\varepsilon}(1+o(1)) .$$

Hence

$$\log M(r, \rho) \sim \frac{(\log r)^{\rho_1^*+1+\varepsilon}}{\rho_1^*+1+\varepsilon}(1+o(1)) ,$$

i.e. $\sigma \leqslant \rho_1^*+1$.

Corollary. *For an entire function* $f(z)$ *of logarithmic order* ρ^* *and logarithmic convergence exponent* ρ_1^*, $\rho^* = \rho_1^*+1$.

Proof. This immediately follows by using theorem 2 in (1.1).

Theorem 3. *If* $f(z)$ *is of logarithmic order* ρ^*, *then*

$$n(r)=O((\log r)^{\rho^*-1+\varepsilon}),$$

for every $\varepsilon>0$.

We omit the proof for conciseness.

Remark Our theorems 1, 2 and 3 are analogous to the results (2.5.8), (2.6.5) and (2.5.12) respectively in [1]. It is to be noted that all our results are not exactly of the same form as for functions of order ρ (compare for instance theorem 2 with (2.6.5) and theorem 3 with (2.5.12) in [1]).

REFERENCES

[1] Boas, R.P: *Entire functions*, Acad. Press, N.Y, 1954.
[2] Shah, S.M. and Ishao, M.: *On the maximum modulus and the coefficients of an entire series*, J. of Indian Math. Soc., 16, (1952), 177-182.

Department of Mathematics,
Hans Raj College,
Delhi University,
Delhi-7, India.

Faculty of Mathematics,
Delhi University,
Delhi, India.