ON CONHARMONICALLY RECURRENT SPACES OF SECOND ORDER

By

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1. A tensor P^h_{ijk} of Riemannian Space V_n is called second order recurrent if

$$P^h_{ijk,lm}=a_{lm}P^h_{ijk}\,, \tag{1.1}$$

for some non-zero tensor a_{lm}, where a comma denotes covariant differentiation with respect to the metric tensor g_{ij}. The tensor a_{ij} of (1.1) is called tensor of recurrence.

A non-flat Riemannian Space is called second order recurrent if the curvature tensor R^h_{ijk} of the space is second order recurrent. A Riemannian space V_n $(n>3)$ is called conformally recurrent space of second order if the conformal curvature tensor C^h_{ijk} given by

$$C^h_{ijk}=R^h_{ijk}-\frac{1}{n-2}(R_k{}^h g_{ij}-R_j{}^h g_{ik}+R_{ij}\delta_k{}^h-R_{ik}\delta_j{}^h)+\frac{R}{(n-1)(n-2)}(\delta_k{}^h g_{ij}-\delta_j{}^h g_{ik}) \tag{1.2}$$

is second order recurrent. Recurrent spaces and conformally recurrent spaces both of second order will be denoted by ${}^2K_n$ and ${}^2C_n$ respectively.

These spaces have been studied by many authors including *Lichnerowicz* [1], *Thompson* [2], *Chaki* and *Roychowdhury* [3]. This paper is concerned with the spaces in which the conharmonic curvature tensor L_{hijk} given by

$$L_{hijk}=R_{hijk}-\frac{1}{n-2}(g_{ij}R_{hk}-g_{hj}R_{ik}+g_{hk}R_{ij}-g_{ik}R_{hj}) \tag{1.3}$$

is second order recurrent. Such spaces will be called conharmonically recurrent-spaces of second order and will be denoted by ${}^2L_n$. ${}^2L_n$'s with some additional conditions are also considered.

2. (i) The scalar curvature and the tensor of recurrence of a ${}^2L_n$

Let L_{hijk} satisfy the relation

(2.1) $$L_{hijk,lm}=a_{lm}L_{hijk}\ ,$$

for some non-zero tensor a_{lm}. From (1.3) and (2.1) it follows that

(2.2) $$R_{hijk,lm}-\frac{1}{n-2}(g_{ij}R_{hk,lm}-g_{hj}R_{ik,lm}+g_{hk}R_{ij,lm}-g_{ik}R_{hj,lm})$$
$$=a_{lm}\left\{R_{hijk}-\frac{1}{n-2}(g_{ij}R_{hk}-g_{hj}R_{ik}+g_{hk}R_{ij}-g_{ik}R_{hj})\right\}.$$

Transvecting (2.2) with g^{ij} and simplifying, we get

$$\frac{1}{n-2}(R_{,lm}-Ra_{lm})g_{hk}=0\ .$$

Therefore, for $n>2$,

(2.3) $$R_{,lm}=a_{lm}R\ .$$

Since a_{lm} is a non-zero tensor, R cannot be a non zero constant. Hence the results:

Theorem 1. *The scalar curvature of a* ^2L_n $(n>2)$ *cannot be a non zero constant.*

Theorem 2. *In a* ^2L_n $(n>2)$ *of non-zero scalar curvature* R*, the tensor of recurrence is symmetric and is given by* $a_{lm}=(1/R)R_{,lm}$.

(ii) The conformal curvature tensor of a ^2L_n

Consider C_{hijk} in a ^2L_n. From (1.2) and (1.3), it follows that

$$C_{hijk}=L_{hijk}+T_{hijk}\ ,$$

where T_{hijk} is given by

$$T_{hijk}=\frac{R}{(n-1)(n-2)}(g_{hk}g_{ij}-g_{hj}g_{ik})\ .$$

Because of (2.3), we have

$$T_{hijk,lm}=a_{lm}T_{hijk}$$

and consequently

$$C_{hijk,lm}=a_{lm}C_{hijk}$$

Hence we have the result:

Theorem 3. *A* ^2L_n $(n>3)$ *is a* ^2C_n *with the same tensor of recurrence.*

(iii) Einstein ^2L_n

For $n>2$, the scalar curvature of an Einstein space is constant. Hence in view of theorem 1, in an Einstein ^2L_n, $R=0$ and consequently $R_{ij}=0$ and L_{hijk} reduces to R_{hijk}. Also in an Einstein ^2K_n, $R_{ij}=0$. Thus we have the following results:

Theorem 4. *The Ricci tensor of an Einstein ^2L_n $(n>2)$ is zero.*

Theorem 5. *A non-flat Einstein space $V_n(n>2)$ is a ^2L_n if and only if it is a ^2K_n with the same tensor of recurrence.*

(iv) Ricci recurrent ^2L_n

A Riemannian space V_n $(n\geqslant 3)$ is called Ricci recurrent spaces of second order if its Ricci tensor satisfies

$$R_{ij,lm}=a^*_{lm}R_{ij} \tag{2.4}$$

and

$$R_{ij}\not\equiv 0\ ,$$

for a non-zero tensor a^*_{lm}. Let us suppose that a ^2L_n is also a Ricci-recurrent space of second order. Transvecting (2.4) with g^{ij}, we get

$$R_{,lm}=a^*_{lm}R\ .$$

Hence, in view of (2.3), we have $a^*_{lm}=a_{lm}$ if $R\neq 0$. From (1.3), it follows that

$$R_{hijk,lm}=L_{hijk,lm}+S_{hijk,lm}\ , \tag{2.5}$$

where S_{hijk} is given by

$$S_{hijk}=\frac{1}{n-2}\left[g_{ij}R_{hk}-g_{hj}R_{ik}+g_{hk}R_{ij}-g_{ik}R_{hj}\right]\ .$$

By virtue of (2.4) and $a^*_{lm}=a_{lm}$, (2.5) reduces to

$$R_{hijk,lm}=a_{lm}(L_{hijk}+S_{hijk})=a_{lm}R_{hijk}\ .$$

Thus we have the result:

Theorem 6. *A second order Ricci recurrent ^2L_n $(n\geqslant 3)$ of non-zero scalar curvature is a ^2K_n with same tensor of recurrence.*

(v) Symmetric ^2L_n

Let a ^2L_n be symmetric. Since it is symmetric, we have

$$R_{hijk,l}=0 \ .$$

Consequently

(2.6) $$R_{hijk,lm}=0 \ ,$$

(2.7) $$R_{ij,lm}=0$$

and

(2.8) $$R_{,lm}=0 \ .$$

Because of (2.3) and (2.8), we get $R=0$. In view of (2.6) and (2.7), from (1.3), it follows that

$$L_{hijk,lm}=0$$

or

(2.9) $$a_{lm}L_{hijk}=0 \ .$$

Since $a_{lm}\neq 0$, (2.9) gives $L_{hijk}=0$.
Also, for $n>2$, $L_{hijk}=0$ implies $R=0$.
Thus we have the result:

Theorem 7. *There are no symmetric* ${}^2L_n$'s ($n>2$) *besides the conharmonically flat ones and the scalar curvature of such spaces is zero.*

(vi) ${}^2L_n$ admitting concurrent vector field

Let a ${}^2L_n$ admit a concurrent vector field ν^i [4]. Then

(2.10) $$\nu^i_{,j}=\rho\delta_j^i \ ,$$

where ρ is a non-zero constant. From (2.10), it follows that

(2.11) $$\nu_{i,j}=\rho g_{ij} \ .$$

Consequently

$$\nu_{i,jk}-\nu_{i,kj}=0 \ .$$

Hence by *Ricci*'s identity

(2.12) $$\nu_h R^h_{ijk}=0 \ .$$

Transvecting (2.12) with g^{ij}_h, we get

(2.13) $$\nu_h R_k{}^h=0 \ .$$

Differentiating (2.13) covariantly with respect to x^l and using (2.11), we have

(2.14) $$\rho R_{lk}+\nu_h R^h_{k,l}=0 .$$

Transvecting (2.14) with g^{lk}, one gets

(2.15) $$2\rho R+\nu^h R_{,h}=0 .$$

Differentiating (2.15) covariantly with respect x^m and using (2.10), we have

(2.16) $$3\rho R_{,m}+\nu^h R_{,hm}=0 .$$

Now transvecting (2.16) with ν^m and applying (2.15), we get

(2.17) $$\nu^h\nu^m R_{,hm}-6\rho^2 R=0 .$$

In consequence of (2.3), (2.17) is reducible to

(2.18) $$(\nu^h\nu^m a_{hm}-6\rho^2)R=0 .$$

If $R\neq 0$, (2.18) gives

(2.19) $$\nu^h\nu^m a_{hm}=6\rho^2 .$$

Hence we have the result:

Theorem 8. *If a* ^2L_n $(n>2)$ *of non-zero scalar curvature admits a concurrent vector field* ν^i *such that* $\nu^i_{;j}=\rho\delta_j^i$, ρ *being a non-zero constant, then* $\nu^h\nu^m a_{hm}=6\rho^2$, *where* a_{ij} *is the tensor of recurrence.*

(vii) ^2L_n admitting parallel vector field

Putting $\rho=0$ in (2.19), one obtains $\nu^h\nu^m a_{hm}=0$.
Thus, we have the result:

Theorem 9. *If a* ^2L_n $(n>2)$ *of non-zero scalar curvature admits a parallel vector field* ν^i, *then* $a_{ij}\nu^i\nu^j=0$, *where* a_{ij} *is the tensor of recurrence.*

(viii) ^2L_n satisfying the condition $R_{ij,k}=R_{ik,j}$

Let a ^2L_n satisfy the condition

(2.20) $$R_{ij,k}-R_{ik,j}=0 .$$

In view of (2.20), we have

(2.21) $$L_{hijk,l}+L_{hikl,j}+L_{hilj,k}=0 .$$

Differentiating (2.21) covariantly with respect to x^m and applying the property of ^2L_n, we get

(2.22) $$a_{lm}L_{hijk}+a_{jm}L_{hikl}+a_{km}L_{hilj}=0 ,$$

a_{ij} being the tensor of recurrence of the space. It can be easily verified that L_{hijk} satisfies the following relations.

(2.23) $$L_{hijk}=-L_{ihjk}=-L_{jkih}=L_{jkhi}=L_{ihkj}\ .$$

Since $a_{ij}\neq 0$, there exists a vector field μ^i such that $a_{ij}\mu^i\mu^j=1$. Let $a_{ji}\mu^i=\alpha_j$. Then $\alpha_i\mu^i=1$. Now put as in Walker [5].

(2.24) $$S_{ij}=-\mu^h\mu^k L_{hijk}=-\mu^h\mu^k L_{jkhi}=-\mu^h\mu^k L_{kjih}=S_{ji}\ .$$

Transvecting (2.22) with $\mu^h\mu^l\mu^m$, we get

$$\mu^h L_{hijk}+(a_{jm}\mu^m)(\mu^h\mu^l L_{hikl})+(a_{km}\mu^m)(\mu^h\mu^l L_{hilj})=0$$

or

(2.25) $$\mu^h L_{hijk}=\alpha_j S_{ik}-\alpha_k S_{ij}\ .$$

Transvecting (2.22) with $\mu^l\mu^m$ and applying (2.25), we have

$$L_{hijk}=S_{hj}\alpha_i\alpha_k+S_{ik}\alpha_h\alpha_j-S_{hk}\alpha_i\alpha_j-S_{ij}\alpha_h\alpha_k\ .$$

Hence we have the result:

Theorem 10. *In a ^2L_n which satisfies (2.20), there exists a vector field α_i such that the tensor L_{hijk} can be expressed in the form*

$$L_{hijk}=S_{hj}\alpha_i\alpha_k+S_{ik}\alpha_h\alpha_j-S_{hk}\alpha_i\alpha_j-S_{ij}\alpha_h\alpha_k\ ,$$

where S_{ij} is a symmetric tensor.

Transvecting (2.20) with g^{ij}, we get

(2.26) $$\frac{1}{2}R_{,k}=0\ .$$

In view of (2.26) and theorem 1, we have $R=0$ and consequently

(2.27) $$a_m{}^h L_{hijk}=0\ .$$

Transvecting (2.22) with g^{ij} and using (2.23) and (2.27), we get

$$a_m{}^h L_{hijk}=0\ ,$$

where $a_m{}^h=g^{ht}a_{tm}$. Hence the result:

Theorem 11. *If a ^2L_n satisfies the condition (2.20), then*

$$a_m{}^h L_{hijk}=0\ ,$$

where $a_m{}^h=g^{ht}a_{tm}$, a_{tm} being the tensor of recurrence.

(ix) Decomposable ^2L_n

Let a ^2L_n $(n>2)$ be a product space $V_p\times V_{n-p}$ whose metric is given by

$$ds^2=\sum_{\alpha,\beta=1}^{n} g_{\alpha\beta}dx^\alpha dx^\beta+\sum_{i,j=p+1}^{n} g_{ij}dx^i dx^j\ , \tag{2.28}$$

where $g_{\alpha\beta}$ and g_{ij} are functions of $(x^1, x^2, \cdots, x^p)$ and $(x^{p+1}, x^{p+2}, \cdots, x^h)$ respectively. We denote by ^1R and ^2R the scalar curvatures of V_p and V_{n-p} respectively. Then the scalar curvature R of ^2L_n is given by

$$R={}^1R+{}^2R\ . \tag{2.29}$$

Since $n>2$, by (2.3) and (2.29), we have

$$({}^1R+{}^2R)_{,lm}=a_{lm}({}^1R+{}^2R)$$

or

$${}^2R_{,lm}=a_{lm}({}^1R+{}^2R)\ , \tag{2.30}$$

where latin indices take the values from $p+1, \cdots, n$.
From (1.3), it follows that

$$\begin{aligned} L_{\alpha i\beta j}&=R_{\alpha i\beta j}-\frac{1}{n-2}[g_{i\beta}R_{\alpha j}-g_{\alpha\beta}R_{ij}+g_{\alpha j}R_{i\beta}-g_{ij}R_{\alpha\beta}]\\ &=\frac{1}{n-2}[g_{\alpha\beta}R_{ij}+g_{ij}R_{\alpha\beta}]\ . \end{aligned} \tag{2.31}$$

From (2.1) and (2.31), we have

$$g_{\alpha\beta}R_{ij,lm}=a_{lm}(g_{\alpha\beta}R_{ij}+g_{ij}R_{\alpha\beta})\ . \tag{2.32}$$

Transvecting (2.32) with g^{ij}, where i and j take values from $p+1, \cdots, n$, we get

$$g_{\alpha\beta}{}^2R_{,lm}=a_{lm}\{g_{\alpha\beta}{}^2R+(n-p)R_{\alpha\beta}\}\ . \tag{2.33}$$

Applying (2.30) in (2.33), we obtain

$$g_{\alpha\beta}a_{lm}{}^1R=a_{lm}(n-p)R_{\alpha\beta}$$

or

$$a_{lm}\{g_{\alpha\beta}{}^1R-(n-p)R_{\alpha\beta}\}=0\ . \tag{2.34}$$

Since $a_{lm}\neq 0$, (2.34) gives

$$R_{\alpha\beta}=\frac{{}^1R}{n-p}g_{\alpha\beta}\ . \tag{2.35}$$

Similarly it can be shown that

$$R_{ij}=\frac{{}^2R}{p}\,g_{ij}\,. \tag{2.36}$$

Thus, we have the theorem:

Theorem 12. *If a* ${}^2L_{2n}$ $(n>1)$ *be a product space* $V_n\times V_n$, *then each of the decomposition spaces and* ${}^2L_{2n}$ *itself are Einstein spaces.*

Transvecting (2.35) with $g^{\alpha\beta}$, where α, β take values from 1, 2, $\cdots$, p, we get

$${}^1R\left(1-\frac{p}{n-p}\right)=0$$

or

$${}^1R(n-2p)=0\,.$$

Similarly from (2.36), one gets

$${}^2R(n-2p)=0\,.$$

If $n\neq 2p$, ${}^1R={}^2R=0$ and hence from (2.29) $R=0$. Thus we have the result:

Theorem 13. *If a* ${}^2L_n$ $(n>2)$ *be a product space* $V_p\times V_{n-p}$, *where* $n\neq 2p$ *then the scalar curvatures of each of decomposition spaces and that of* ${}^2L_n$ *itself are zero.*

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