SOME VARIANTS OF A STRICT-SET-CONTRACTION

By

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1. Introduction.

Let A be any bounded subset of a Banach space X. *Darbo* [1] defined $\gamma(A)$, the measure of noncompactness of A, to be $\inf\{d>0|A$ can be covered by a finite number of sets of diameter less than or equal to $d\}$.

If D is a subset of the Banach space X and f is a continuous map from D to X, f is called a k-set-contraction, if $\gamma[f(A)] \leqq k\gamma[A]$ for A, any bounded subset of D. Specially if $k<1$, we say that f is a strict-set-contraction, an important example of which is furnished by a map of the form $U+T$, U is a strict contraction (*i.e.* $\|Ux+Uy\|\leqq k\|x-y\|$, $k<1$) and T is completely continuous (not necessarily linear).

Darbo proved that if D is a bounded closed convex subset of a Banach space X and $f: D\rightarrow D$ is a strict-set-contraction, then f has a fixed point.

In this paper, we consider a map of the form $U+T$, U is a linear bounded iteratively strict-set-contraction (*i.e.* a linear bounded operator such that some iterate U^p is a strict-set-contraction) and T is completely continuous (not necessarily linear).

2. Background.

At first we mention the fundamental properties of the measure of noncompactness in the form of the proposition. They are useful to prove theorems latter.

Proposition 1. *Let X be a Banach space and A and B are bounded subsets of X. Then we have*

(*a*) *if* $A\subset B$, *then* $\gamma(A)\leqq\gamma(B)$,

(*b*) $\gamma(A\cup B)=\max\{\gamma(A), \gamma(B)\}$,

(*c*) $\gamma(\bar{A})=\gamma(A)$ *if* $\bar{A}$ *denotes the closure of* A,

(*d*) $\gamma(\overline{co}A)=\gamma(A)$ *if we denote the convex closure of A by* $\overline{co}\ A$,

(*e*) $\gamma(A+B)\leqq\gamma(A)+\gamma(B)$ *if we denote* $\{a+b|a\in A, b\in B\}$ *by* $A+B$, (*f*) *if A is compact, then $\gamma(A)=0$.*

Proposition 2. *Let B be a closed bounded convex set in a Banach space X, and let $f:B\to B$ be a continuous map. Let $B_1=\overline{co}f(B)$, $B_n=\overline{co}f(B_{n-1})$, for $n>0$. Assume that $\gamma(B_n)\to 0$. Then f has a fixed point.*

The proofs of these propositions are contained in the references [1] and [2].

3. Fixed point theorem.

Theorem 1. *Let B be an open ball of radius r and center θ, origin of a Banach space X. If $f:X\to X$ is a map of the form $U+T$, U is a linear bounded iteratively strict-set-contraction and T is completely continuous, and f satisfies the boundary condition*

$$(LS):\ f(x)=\alpha x \text{ for some } x \text{ in } \partial B, \text{ then } \alpha\leqq 1$$

where ∂B denotes the boundary of B. Then f has a fixed point in $\bar{B}$.

In the proof of theorem 1 we shall make use of the following lemmas.

Lemma 1. *Let D be any bounded subset of a Banach space X. If $f:X\to X$ is a map of the form $L+T$, L is a linear bounded map and T is completely continuous. Then we have $\gamma[f(\overline{co}(D\cup\theta))]=\gamma[f(D)]$.*

Proof. Since $\overline{co}(D\cup\theta)\supset D$, by Proposition 1($a$), $\gamma[f(\overline{co}(D\cup\theta))]\geqq\gamma[f(D)]$. On the other hand, since $f(\overline{co}(D\cup\theta))\subset L(\overline{co}(D\cup\theta))+T(\overline{co}(D\cup\theta))$ and the compactness of the map T, we see that $\gamma[f(\overline{co}(D\cup\theta))]\leqq\gamma[L(\overline{co}(D\cup\theta))]$ by Proposition 1 (e), (f). Since L is linear, it follows that $L(\overline{co}(D\cup\theta))=\overline{co}(L(D\cup\theta))=\overline{co}(L(D)\cup L(\theta))$, so $\gamma[L(\overline{co}(D\cup\theta))]=\gamma[L(D)]$. By Proposition 1 (e), (f), $\gamma[L(D)]\leqq\gamma[f(D)]+\gamma[-T(D)]\leqq\gamma[f(D)]$. Hence we have that $\gamma[f(\overline{co}(D\cup\theta))]\leqq\gamma[f(D)]$ and consequently $\gamma[f(\overline{co}(D\cup\theta))]=\gamma[f(D)]$.

Lemma 2. *Let R be the radial retraction of X onto $\bar{B}$, i.e.*

$$R(x)=\begin{cases} x \text{ if } \|x\|\leqq r\,, \\ (rx/\|x\|) \text{ if } \|x\|\geqq r\,. \end{cases}$$

Let $f:X\to X$ be a map of the form $U+T$, U is a linear bounded iteratively strict-set-contraction and T is completely continuous, and we define the map $F(x)=R(f(x))$ for all x in $\bar{B}$. Then $F:\bar{B}\to\bar{B}$ has a fixed point.

Proof. Let $B_1=\overline{co}F(\bar{B})$, $B_{n+1}=\overline{co}F(B_n)$ and let $C_1=\overline{co}(f(\bar{B})\cup\theta)$, $C_{n+1}=\overline{co}(f(C_n)\cup\theta)$ for $n>0$. Clearly $B_{n+1}\subset B_n$ for $n>0$ and since $R(D)\subset\overline{co}(D\cup\theta)$ for

D, any bounded subset of X, then we have $B_n \subset C_n$ for $n>0$.

Therefore $\liminf_{n\to\infty} \gamma(C_n)$ implies that $\gamma(B_n)\to 0$. Since $\bar{B}$ is a closed bounded convex subset of X, in order to prove this lemma by Proposition 2, it suffices to show that $\liminf_{n\to\infty} \gamma(C_n)=0$. Since f^j is a map of the form U^j+T_j, where T_j is completely continuous and U^j is a linear bounded map, $\gamma[f^j(\overline{\text{co}}(D\cup\theta))]=\gamma[f^j(D)]$ follows from Lemma 1.

Applying this relation repeatedly we see

$$\gamma(C_i)=\gamma[\overline{\text{co}}(f(C_{i-1})\cup\theta)]=\gamma[f(C_{i-1})]=\gamma[f(\overline{\text{co}}(f(C_{i-2})\cup\theta))]$$
$$=\gamma[f^2(C_{i-2})]=\cdots=\gamma[f^{i-1}(C_1)]=\gamma[f^i(\bar{B})]\,.$$

There is some integer $p>0$ such that U^p is a k-set-contraction, $k<1$, since U is an iteratively strict-set-contraction.

If $i=pn$ $(n>0)$, then we have

$$\gamma(C_{pn})=\gamma[f^{pn}(\bar{B})]\leqq\gamma[U^{pn}(\bar{B})]=\gamma[T_{pn}(\bar{B})]\leqq\gamma[U^{pn}(\bar{B})]\leqq k^n\gamma(\bar{B})\,.$$

Therefore this implies $\liminf_{n\to\infty} \gamma(C_n)=0$.

Proof of theorem 1. By lemma 2, there exists $u\in\bar{B}$ such that $F(u)=u$. But then u is also a fixed point of f. Indeed if $u\in B$, then $\|R(f(u))\|<r$. Therefore $R(f(u))=f(u)=u$. Alternatively, if $u\in\partial B$ and u is not a fixed point of f, then $\alpha=\|f(u)\|/r>1$, which is excluded by our condition (LS). Hence u is a fixed point of f.

Corollary 1. *Let $f:X\to X$ be a map of the form $U+T$, U is a linear bounded iteratively strict-set-contraction and T is completely continuous, and suppose that f satisfies any one of the following conditions*:

(*a*) $f(\bar{B})\subset\bar{B}$,

(*b*) $f(\partial B)\subset\bar{B}$,

(*c*) $\|f(x)-x\|^2\geqq\|f(x)\|^2-\|x\|^2$, *for all x in ∂B*,

(*d*) $(f(x),\ \omega)\leqq(x,\ \omega)$, *any $\omega\in J(x)$, for all x in ∂B, where J is a duality mapping of X into the set of all subsets of X^* such that*

$$J(x)=\{\omega|\omega\in X^*;\ \|\omega\|=\|x\|;\ (x,\ \omega)=\|x\|\cdot\|\omega\|\}\,.$$

Then f has a fixed point in $\bar{B}$.

Proof. Clearly (a) and (b), each separately, implies (LS). Hence the theorem 1 is applicable. Next suppose that $f(x)=\alpha x$ for some x in ∂B. Then (c) implies that $(\alpha-1)^2\geqq\alpha^2-1$. So $\alpha\leqq1$. And (d) implies that $(\alpha x,\omega)\leqq(x,\omega)$.

So $\alpha\leqq 1$. So $\alpha\leqq 1$. Therefore (c) and (d) respectively implies (LS). Hence the theorem 1 is applicable.

4. Mapping theorem.

A map f is said to be quasi-bounded if the number defined by

$$|f|=\limsup_{\|x\|\to\infty}\frac{\|f(x)\|}{\|x\|},$$

is finite. The number $|f|$ is called the quasi-norm of f. It is easy to see that f is quasi-bounded if and only if there exist positive constants α and β such that $\|x\|\geqq\alpha$ implies that $\|f(x)\|\leqq\beta\|x\|$.

Theorem 2. *Let $f:X\to X$ be a map of the form $U+T$, U is a linear bounded iteratively strict-set-contraction and T is completely continuous, and let f be quasi-bounded with $|f|<1$. Then any given y in X, there exists x in X such that $x-f(x)=y$.*

Proof. Let $\varepsilon>0$ be such that $|f|+\varepsilon<1$. Since f is quasi-bounded, if $\|x\|$ is large enough, we see

$$\|f(x)\|\leqq(|f|+\varepsilon)\|x\| .$$

Let y be any element of X, on the assumption that $\|x\|$ is large enough, we see

$$\begin{aligned}
&\|f(x)+y-x\|^2+\|x\|^2-\|f(x)+y\|^2\\
\geqq&\|x\|^2-(\|f(x)\|+\|y\|)^2\\
\geqq&\|x\|^2-\|f(x)\|^2-2\|y\|\cdot\|f(x)\|-\|y\|^2\\
\geqq&(1-(|f|+\varepsilon)^2)\|x\|^2-2\|y\|\cdot(|f|+\varepsilon)\|x\|-\|y\|^2 .
\end{aligned}$$

Therefore there exists large number r such that

$\|f(x)+y-x\|^2\geqq\|f(x)+y\|^2-\|x\|^2$ for all x such that $\|x\|=r$. Since the map $f':X\to X$ defined by $f'(x)=f(x)+y$ is easily seen to be a map of the form $U+T+y$, U is a linear bounded iteratively strict-set-contraction and $T+y$ is completely continuous, then Corollary 1 (c) is applicable for f'. Therefore given any y in X, there exists x in X such that $f'(x)=f(x)+y=x$.

Corollary 2. *If V be a bounded linear operator on X such that the iterate V^p is a strict contraction for some $p>0$ and T be quasi-bounded and completely continuous on X. If in addition the quasi-norm of T satisfies*

$$|T|<1-k$$

where $k=\sup_{\|x\|\leq 1}\|V(x)\|$. *Then for any y in X, there exists x in X such that* $x-V(x)-T(x)=y$.

This corollary was proved by *Nashed and Wong* [3]. But this is a special case of our theorem 2.

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REFERENCE

[1] G. Darbo, *Punti uniti in transformazioni a condominio non compatto*, Rend. Sem. Mat. Univ. Padova 24 (1955), 84-92.

[2] R.D. Nussbaum, *The fixed point index for local condensing maps*, Ann. Mat. Pura. Appl. 89 (1971) 217-258.

[3] M.Z. Nashed and J.S. Wong, *Some variants of a fixed point theorem of Krasnoselskii and applications to nonlinear integral equations*. J. Math. Mech. 18 (1969) 767-777.

[4] D.E. Edmunds and J.R.L. Webb, *Nonlinear operator equations in Hilbert space*, J. Math. Anal. Appl. 34 (1971) 471-478.

[5] W.V. Petryshyn, *Structure of the fixed point set of k-set-contraction*, Arch. Rational Mech. Anal. 40 (1971) 312-328.

[6] _______, *A new fixed point theorem and its application*, Bull. Amer. Math. Soc. 78 (1972) 225-229.

[7] B.N. Sadovskii, *On a fixed point principle*, Function. Analiz. Prilozen. 1 (1967) 74-76 (Russian).

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