$\tilde{L}$-EQUIVALENCE AND REPRESENTATIONS OF HOMOLOGY CLASSES

By

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§ 0. Introduction

In the paper we will discuss on an equivalence relation between $(n-1)$-submanifolds in an n-manifold, and on representations of $(n-1)$-dimensional homology classes of an n-manifold. We shall work in the PL category. By the smoothing theory, required results in the differentiable category may be obtained.

R. Thom [8] introduced the concept of "*L-equivalence*" between submanifolds in a manifold. We shall introduce the concepts of "$\tilde{L}$-equivalence" between $(n-1)$-submanifolds in an n-manifold, and of "$\tilde{L}$-manifold", see Definitions 4 and 5 in § 2. Then, we shall obtain;

Theorem 1. *Every compact connected orientable manifold, without boundary or with connected boundary, is an $\tilde{L}$-manifold.*

It is a well known result by *R. Thom* [8] that every $(n-1)$-dimensional homology class $\theta \in H_{n-1}(M; Z)$ of an orientable n-manifold M is representable by a submanifold in M. But, he didn't refer to the number of connected components of the submanifold representing θ. We will give the necessary and sufficient condition for θ to be representable by a submanifold having β connected components.

Let M be a compact orientable n-manifold. By the Poincaré duality, $H_{n-1}(M; Z)$ is a free abelian group of finite rank. Let $\{g_1, g_2, \cdots, g_r\}$ be a free abelian basis for $H_{n-1}(M; Z)$. An implication of Theorem 1 is as follows;

Theorem 2. *Let M be a compact connected orientable n-manifold without boundary or with connected boundary. For a non-trivial homology class*

$$\theta = a_1 g_1 + a_2 g_2 + \cdots + a_r g_r ,$$

of $H_{n-1}(M; Z)$ the following conditions are equivalent;

(1) *θ can be represented by a submanifold having β connected components in M.*

(2) $\beta \geqq |\alpha|$, *where* α *is the greatest common divisor* $(a_1, a_2, \cdots, a_r)$ *of* $a_1, a_2, \cdots, a_r$.

So, as a special case of Theorem 2, we have;

Theorem 3. *Let* M *be a compact connected orientable* n*-manifold without boundary or with connected boundary. For a non-trivial homology class*

$$\theta = a_1 g_1 + a_2 g_2 + \cdots + a_r g_r ,$$

of $H_{n-1}(M; Z)$, *the following conditions are equivalent;*

(1) θ *can be represented by a connected submanifold in* M.

(2) *the greatest common divisor* $(a_1, a_2, \cdots, a_r) = 1$.

First in this direction, *T. Kaneko* [2] obtained a complete answer to this problem for closed surfaces in 1965. Then *S. Suzuki* extended his result to surfaces with boundary [7] and also in the direction for the above Theorem 2 [6]. The author gave an extension of our problem to 3-manifolds under some conditions [9]. In this paper this problem will be perfected. In preparing this paper, *M. Kato* has informed the author that independently *H. Nakatsuka* has also proved our Theorem 3 by different methods.

In § 1, we shall introduce an elementary operation $\square_k$, and give the proof of Lemma 1, which is of importance in the sequel. In § 2, we shall introduce the concepts of "$\tilde{L}$-equivalence" and "$\tilde{L}$-manifold". We shall consider the relation between "L-equivalence" and "$\tilde{L}$-equivalence", and prove Theorem 1. In § 3, we will give the proofs of Theorems 2 and 3.

In this paper, all manifolds will be compact and oriented. PL embeddings will be locally flat and submanifolds in a manifold will be locally flat and closed. And an ambient n-manifold M is always connected, but a submanifold in a manifold is not always connected unless we mention.

I am indebted to Professors *T. Homma*, *F. Hosokawa*, *M. Kato*, *S. Suzuki* and *T. Yanagawa* for their help.

§ 1. Notations and operations *k*

Throughout the paper, ∂M, $intM$, clM denote the boundary, the interior and the closure of a manifold M, respectively. $N(A; B)$ denotes a regular neighborhood of a subpolyhedron A in a polyhedron B. Homeomorphism and isomorphism are denoted by the same symbol $\cong$, while $\approx$, $\simeq$ and $\sim$ denote, respectively, to isotopy, homotopy and homology. I, D^k and Δ^m mean, respectively, the closed interval $[0, 1]$, k-simplex and m-simplex.

For a submanifold A in a manifold, $-A$ is the submanifold having the opposite orientation of A. For integers $a_1, a_2, \cdots, a_\lambda$, $(a_1, a_2 \cdots . a_\lambda)$ denotes the greatest common divisor of them. Especially, we assume that $(a_1, a_2, \cdots, a_\lambda) \geqq 0$.

Let A be an $(n-1)$-submanifold in an n-manifold M. Let l be a simple oriented arc in M that spans A with discoherent orientation, *i.e.* $l \cap A = \partial l \cap A = \partial l$, and let $f: I \longrightarrow M$ be an embedding such that $f(I) = l$. Then, we can add an n-cell $I \times \Delta^{n-1}$ to A by a homeomorphism $\bar{f}: I \times \Delta \longrightarrow N(l; M)$ such that $\bar{f}|_{I \times \{v\}} = f$, $\bar{f}(\partial I \times \Delta) \subset A$ and $\bar{f}(int I \times \Delta) = \phi$; and obtain an $(n-1)$-submanifold $A_* = \{A - \bar{f}(\partial I \times \Delta)\} \cup \bar{f}(I \times \partial \Delta)$ having the orientation induced by A. When we perform the operation above, we shall say that the $(n-1)$-submanifold A_* is obtained from A by an *operation* $\square_1$ (along l in M) and denote $A \square_1$. For any integer k, let $f: D^k \times \Delta^{m-k} \longrightarrow M$ be an embedding such that $f(\partial D \times \Delta) \subset A$ and $f(int D \times \Delta) \cap A = \phi$. We can add an n-cell $D^k \times \Delta^{m-k}$ to A by an embedding f and obtain an $(n-1)$-submanifold $A_* = \{A - f(\partial D \times \Delta)\} \cup f(D \times \partial \Delta)$ having the orientation induced by A. Then, we shall say that the $(n-1)$-submanifold A_* is obtained from A by an *operation* $\square_k$ and denote $A \square_k$. Moreover, we will denote $A \square_1 \square_2 \square_1 \square_1 \square_2$ and $A \square_3 \square_3 \square_2 \square_1 \square_2$ etc. by $A \square_1^3 \square_2^2 \square_3^0 \cdots \square_n^0$ and $A \square_1^1 \square_2^2 \square_3^2 \square_4^0 \cdots \square_n^0$ etc. unless confusion. It will be noticed that;

Proposition 1. *An operation $\square_k$ does not exchange the homology class, that is $A = A \square_k$ in $H_{n-1}(M; Z)$.*

In general for two submanifolds A and B in M, we cannot always apply an operation $\square_1$, see Lemma 2.

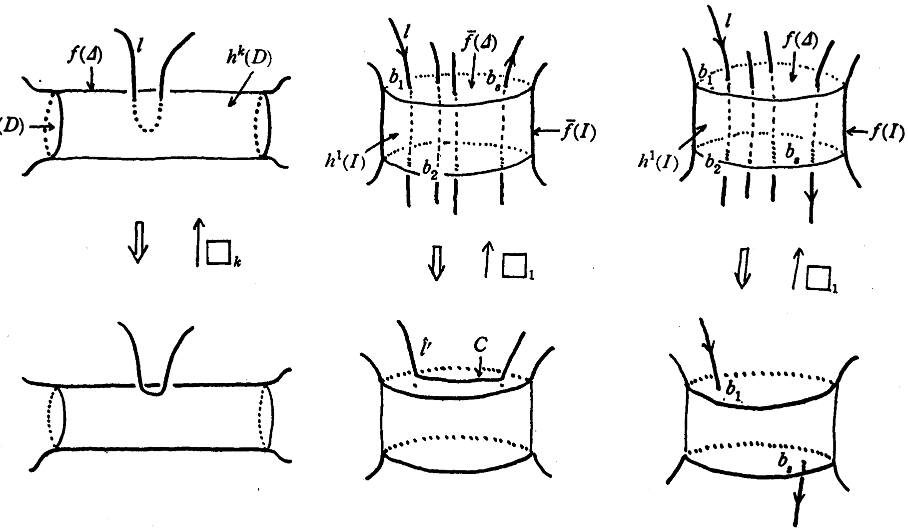
Definition 1. Let A be non-connected $(n-1)$-submanifold in an n-manifold M and let A_1, A_2 be distinct connected components of A. Then, A_1 and A_2 are said to be *well-situated* (*relative to* A) iff there is a simple oriented arc l that spans A_1 and A_2 with discoherent orientation, *i.e.* $l \cap A = l \cap (A_1 \cup A_2) = \partial l \cap (A_1 \cup A_2) = \partial l$.

Lemma 1. *Let A_1, A_2 be disjoint connected $(n-1)$-submanifold in an n-manifold M. If A_1 and A_2 are not well-situated relative to $A_1 \cup A_2$, then any one component of $A_1 \square_1^{\lambda_1} \square_2^{\lambda_2} \square_3^{\lambda_3} \cdots \square_{n-1}^{\lambda_{n-1}}$ and any one component of $A_2 \square_1^{\mu_1} \square_2^{\mu_2} \square_3^{\mu_3} \cdots \square_{n-1}^{\mu_{n-1}}$ are not well-situated relative to $A_1 \square_1^{\lambda_1} \square_2^{\lambda_2} \square_3^{\mu_3} \cdots \square_{n-1}^{\lambda_{n-1}} \cup A_2 \square_1^{\mu_1} \square_2^{\mu_2} \square_3^{\mu_2} \cdots \square_{n-1}^{\mu_{n-1}}$.*

Proof. Suppose that a component, say B_1, of $A_1 \square_1^{\lambda_1} \square_2^{\lambda_2} \square_3^{\lambda_3} \cdots \square_{n-1}^{\lambda_{n-1}}$ and a component, say B_2, of $A_2 \square_1^{\mu_1} \square_2^{\mu_2} \square_3^{\mu_3} \cdots \square_{n-1}^{\mu_{n-1}}$ are well-situated relative to

$A_1\square_1^{\lambda_1}\square_2^{\lambda_2}\square_3^{\lambda_3}\cdots\square_{n-1}^{\lambda_{n-1}}\cup A_2\square_1^{\mu_1}\square_2^{\mu_2}\square_3^{\mu_3}\cdots\square_{n-1}^{\mu_{n-1}}$. So there is a simple oriented arc l that spans B_1 and B_2 with discoherent orientation. We may assume ∂l, say a_1, a_2, is contained in A_1 and A_2 by slight deformation.

Let $\nu=\lambda_1+\lambda_2+\cdots+\lambda_{n-1}+\mu_1+\mu_2+\cdots+\mu_{n-1}$. If the ν-th operation is an operation $\square_k$, $k=2, 3, \cdots, n-1$, we can exclude the intersection of l and the k-handle $h^k(D)=f(D^k\times\Delta^{n-k})$ by deforming l isotopically, since $l\cap h^k(D)=$arcs, $l\cap\partial h^k(D)=l\cap(\partial D\times\Delta)$ and $\partial D\times\Delta\cong S^{k-1}\times\Delta^{n-k}$, see Fig. 1(*a*). If the ν-th operation is an operation $\square_1$, we deform l isotopically so that l intersects with $f(\partial I\times\Delta^{n-1})$ transversally. If $l\cap f(I\times\Delta^{n-1})\neq\phi$, then it consists of some arcs, and $l\cap f(\partial I\times\Delta)$ consists of some points $b_1, b_2, \cdots, b_s$ numbered from a_1 to a_2 on l. If b_1 and b_s are contained in same component, say C, of $f(\partial I\times\Delta)$, we can join b_1 and b_s in C and obtain a simple arc $l'=\overline{a_1b_1}\cup\overline{b_1b_s}\cup\overline{b_sa_2}$. By deforming l' isotopically, we can obtain a simple arc $\bar{l}'$ such that $\bar{l}'\cap f(I\times\Delta)=\phi$, see Fig. 1(*b*). So, we may assume b_1 and b_s are not contained in same component of $f(\partial I\times\Delta)$. Then we will reserve only two simple arcs $l_1'=\overline{a_1b_1}$ and $l_2'=\overline{b_sa_2}$, see Fig. 1 (*c*).



(a) (b) (c)

Fig. 1.

By the repetition of the procedure, we have simple oriented subarcs $l_1, l_2, \cdots, l_p$ of l such that $l_i\cap(A_1\cup A_2)=\partial l_i\cap(A_1\cup A_2)=$two points, say c_i, d_i. Clearly, d_i and c_{i+1} are contained in A_j, and homology insersection numbers at the points d_i, c_{i+1} are 1 and -1 (or -1 and 1), $i=1, 2, \cdots, p-1$, $j=1, 2$, see Fig. 1 (*b*) (*c*). Since A_1, A_2 is connected, we can join d_i and c_{i+1} by simple arc $\bar{l}_i$ and obtain a simple oriented arc $\bar{l}=l_1\cup\bar{l}_1\cup l_2\cup\bar{l}_2\cup\cdots\cup\bar{l}_{p-1}\cup l_p$. By deforming

$\bar{l}$ isotopically, we can obtain a simple oriented arc $\hat{l}$ that spans A_1 and A_2 with discoherent orientation. Therefore, A_1 and A_2 are well-situated, which is a contradiction.

Lemma 2. *Let M be an n-manifold with non-connected boundary. Let A_1, A_2 be distinct component of ∂M having the orientation induced by M. Then, A_1 and $(-A_2)$ are not well-situated.*

Proof. Let l be a simple arc in M that spans A_1 and A_2. If we give an arbitrary orientation for l, then it has the coherent orientation with A_1 (or $-A_2$), as M is orientable. And if we give the another orientation for l, then it has the coherent orientation with $-A_2$ (or A_1). That is, there exists no arc in M oriented discoherently with A_1 and $(-A_2)$.

§ 2. *L*-equivalence and $\tilde{L}$-equivalence

Definiton 2. (*R. Thom* [8]) Let U, V be m-submanifolds in an n-manifold M. Then, we say that U and V are *L-equivalent*, iff there is an $(m+1)$-submanifold W in $M\times I$ satisfying the following conditions;

(1) $W\cap M\times\{0\}=U$, $\quad W\cap M\times\{1\}=V$;

and

(2) $\partial W=U-V$.

Definition 3. An n-manifold M is said to be an *L-manifold*, iff for any two $(n-1)$-submanifolds U, V such that $U\sim V$, U and V are L-equivalent.

Proposition 2. (*R. Thom* [8]) *Every manifold is an L-manifold.*

We will introduce stronger concepts than these as follows;

Definition 4. Let U, V be $(n-1)$-submanifolds in an n-manifold M. Then, we say that U and V are $\tilde{L}$*-equivalent*, iff there exists a sequence $\{A_1, A_2, \cdots, A_s\}$ of $(n-1)$-submanifolds in M so that

(1) $U=A_1$, $\quad V=A_s$;

and

(2) A_{i+1} is isotopic to A_i or obtained from A_i by an operation $\square_k$; $i=1, 2, \cdots, s-1$, $k=1, 2, \cdots, n-1$.

Definition 5. An n-manifold M is said to be an $\tilde{L}$*-manifold*, iff either one of the following conditions is satisfied;

(1) every $(n-1)$-submanifold is homologous to 0 in M;

or

(2) for any two $(n-1)$-submanifolds U, V such that $U, V \not\sim 0$ and $U \sim V$, U and V are $\tilde{L}$-equivalent

Let U be an $(n-1)$-submanifold in an n-manifold M without boundary or with connected boundary. And let $S_1, S_2 \cdots, S_r$ be mutually disjoint $(n-1)$-spheres such that each S_i bounds n-ball B_i in M and $B_i \cap U = \phi$.

We apply an operation $\square_1$ for distinct components of $\tilde{U} = U \cup S_1 \cup S_2 \cup \cdots \cup S_r$, denoted $\tilde{U} \xrightarrow{1} \tilde{U}\square_1$, if it is possible. If we can still apply an operation $\square_1$ to distinct components of $\tilde{U}\square_1$, we apply an operation $\square_1$. We repeat this procedure as often as possible, and have an $(n-1)$-submanifold $\tilde{U}\square_1^\lambda$ such that we cannot apply an operation $\square_1$ to distinct components of $\tilde{U}\square_1^\lambda$. Then, we have the following diagram;

$$(*)\quad \begin{array}{ccccccccc} \tilde{U} & \xrightarrow{1} & \tilde{U}\square_1 & \xrightarrow{1} \cdots & \xrightarrow{1} & \tilde{U}\square_1^i & \xrightarrow{1} \cdots & \xrightarrow{1} & \tilde{U}\square_1^\lambda \\ \cup & & \cup & & & \cup & & & \cup \\ U_0 = U & \longrightarrow & U_1 & \longrightarrow \cdots & \longrightarrow & U_i & \longrightarrow \cdots & \longrightarrow & U_\lambda \end{array}$$

where U_{i+1} is an $(n-1)$-submanifold obtained from U_i by applying the operation $\square_1$ in the following sense;

(1) $U_{i+1} = U_i$, when the i-th operation $\square_1$ is applied to components of $\tilde{U}\square_1^i - U_i$.

(2) $U_{i+1} = (U_i \cup A)\square_1$, when the i-th operation $\square_1$ is applied to U_i and a component A of $\tilde{U}\square_1^i - U_i$.

(3) $U_{i+1} = U_i\square_1$, when the i-th operation $\square_1$ is applied to U_i.

Lemma 3. *If $U_\lambda \neq \tilde{U}\square_1^\lambda$, then U is homologous to* 0 *in M.*

Proof. Suppose $U_\lambda \neq \tilde{U}\square_1^\lambda$. So there exists a component S^{n-1} of $\tilde{U}\square_1^\lambda - U_\lambda$. Clearly, S^{n-1} bounds n-ball B^n. Let C_1 be a component of $M - N(\tilde{U}\square_1^\lambda; M) - B$ whose boundary contains S^{n-1}. Let $T_1, \cdots, T_p$ be components distinguished S^{n-1} of ∂C^1 such that $T_i \subset \tilde{U}\square_1^\lambda$. By assumption, if S^{n-1} has discoherent (or coherent) orientation with C_1, T_i has coherent (or discoherent) orientation with C_1, $i = 1, 2, \cdots, p$. If $p > 1$ then T_1 and T_2 are well-situated which is a contradiction to the property of $\tilde{U}\square_1^\lambda$. Hence $p = 1$ and T_1 is homologous to 0 since ∂M is homologous to 0, see Fig. 2.

Let C_2 be a component of $M - N(\tilde{U}\square_1^\lambda; M) - B - C_1$ whose boundary contains T_1. By the repetition of the above procedure, we have a component T_2' of $\tilde{U}\square_1^\lambda - B$ homologous to 0.

We repeat the above procedure as often as possible, and have a sequence $\{T_1, T_2', \cdots, T_{l-1}'\}$ of all components of $\tilde{U}\square_1^\lambda - B$. And every component of $\tilde{U}\square_1^\lambda - B$ is homologous to 0 in M.

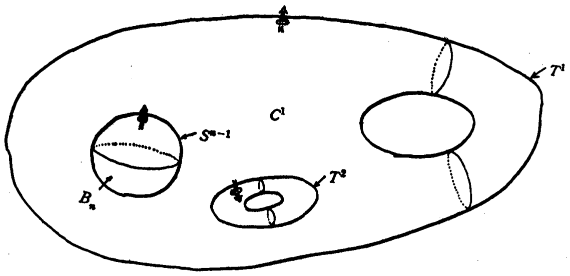


Fig. 2.

Since every component of $\tilde{U}\square_1^\lambda$ in B is homologous to 0, $\tilde{U}\square_1^\lambda$ is homologous to 0. Therefore U is homologous to 0 in M.

Lemma 4. *M, U, S_i, $\tilde{U}$, $\tilde{U}\square_1^\lambda$; the same as above. If U is not homologous to 0 in M, then U and $\tilde{U}$ are $\tilde{L}$-equivalent.*

Proof. It is enough to prove that U and $\tilde{U}\square_1^\lambda$ are $\tilde{L}$-equivalent. We will proceed by induction on λ. If $\lambda=0$, by Lemma 3, $r=0$ and Lemma is obvious. Suppose $\lambda>0$. We have the diagramm (‡), and moreover $\tilde{U}\square_1^\lambda=U_\lambda$ by Lemma 3. By induction then, U_1 and $\tilde{U}\square_1^\lambda$ are $\tilde{L}$-equivalent. Therefore it is enough to prove that U and U_1 are $\tilde{L}$-equivalent. If the first operation $\square_1$ falls under the case (1) or (3), then it is obvious. Suppose the first operation $\square_1$ falls under the case (2). That is, the first operation $\square_1$ is applied to U and S_i, $i=1, 2, \cdots, r$. Since S_i bounds n-ball B_i in M and $B_i \cap U=\phi$, U is isotopic to $U_1=(U\cup S_i)\square_1$. Therefore U and $\tilde{U}\square_1^\lambda$ are $\tilde{L}$-equivalent.

Lemma 5. *Every L-manifold, without boundary or with connected boundary, is an $\tilde{L}$-manifold.*

Proof. If every $(n-1)$-submanifold in M is homologous to 0, Lemma is obvious. So we may assume that there is an $(n-1)$-submanifold, say U, in M which is not homologous to 0.

Let V be an $(n-1)$ submanifold in M such that $V\sim U$. By assumption, there is an n-submanifold W in $M\times I$ satisfying the following conditions;

(1) $W\cap M\times\{0\}=U$, $\quad W\cap M\times\{1\}=V$;

and

(2) $\partial W=U-V$.

By deforming W isotopically, we have an n-submanifold W' in $M\times I$ such that $f|_{W'}\colon W'\longrightarrow I$ is a non-degenerate mapping, see [3] [4], where $f\colon M\times I\longrightarrow I$ is a projection. Since the critical points are finite, we may assume that there exist no critical points in $M\times[0, 1/4]\cup M\times[3/4, 1]$. By deforming W' isotopically, we have an n-submanifold $\tilde{W}$ in $M\times I$ satisfying the following conditions;

(1) $f|_{\tilde{W}}: \tilde{W}\longrightarrow I$ is non-degenerate,

(2) the critical points of index 0 consist in $M\times(0,1/4)$,

(3) the critical points of index n consist in $M\times(3/4,1)$,

and

(4) the critical points of index $1,2,\cdots,n-1$ consist in $M\times(1/4,3/4)$.

Therefore, $M\times\{1/4\}$ consists of an $(n-1)$-submanifold U' and $(n-1)$-spheres $S_1,S_2,\cdots,S_r$ such that U' is isotopic to U in M and each sphere S_i bounds n-ball B_i in M; and $M\times\{3/4\}$ consists of an $(n-1)$-submanifold V' and $(n-1)$-spheres $S_1',S_2',\cdots,S_t'$ such that V' is isotopic to V in M and each sphere S_j' bounds n-ball B_j' in M. Clearly $B_i\cap U'=\phi$ and $B_i'\cap V'=\phi$.

Since the critical point of index k corresponds the operation $\Box_k$, $k=1,2,\cdots,n-1$, $\tilde{U}=U'\cup S_1\cup S_2\cup\cdots\cup S_r$ and $\tilde{V}=V'\cup S_1'\cup S_2'\cup\cdots\cup S_t'$ are $\tilde{L}$-equivalent in M. By Lemma 4, $\tilde{U}$ and U' are $\tilde{L}$-equivalent, and $\tilde{V}$ and V' are $\tilde{L}$-equivalent. Hence U' and V' are $\tilde{L}$-equivalent. Therefore, U and V are $\tilde{L}$-equivalent.

Combining this with Proposition 2, we will obtain the following;

Theorem 1. *Every connected manifold, without boundary or with connected boundary, is an $\tilde{L}$-manifold.*

While, for the other manifolds, we will obtain the following;

Proposition 3. *Every manifold M with non-connected boundary is not an $\tilde{L}$-manifold.*

Proof. Let $f:D^n\longrightarrow intM$ be an embedding. Then, $S=f(\partial D)$ is an $(n-1)$-sphere in M. Let A be a component of ∂M. Since $M-f(D)$ is an orientable n-manifold, we give the orientation induced from $M-f(D)$ to A and S. By Lemma 2, A and $-S$ are not well-situated. Therefore, A and $A\cup(-S)$ are not $\tilde{L}$-equivalent. While, A is homologous to $A\cup(-S)$ and $A\not\sim 0$. Hence M is not an $\tilde{L}$-manifold.

§ 3. Proof of Theorem 2

[Proof of Theorem 3 (2)$\longrightarrow$(1).]

According to [8] p. 55, Théorème II. 27, there is an $(n-1)$-submanifold representing θ in M. Let A be an $(n-1)$-submanifold such that the number of connected components of A is smallest in $(n-1)$-submanifolds representing θ in M. Clearly, there is no component of A homologous to 0 in M, and we cannot apply an operation $\Box_1$ for distinct components of A.

Let $C_1, C_2, \cdots, C_p$ be connected components of A. Suppose $p>1$. Let M_i be a component of $M-N(A; M)$ and let $V_1, V_2, \cdots, V_{t(i)}$ be components of $\partial M_i - \partial M$. Clearly $t(i) \geqq 1$. Since $V_j \sim 0$ in M for any j, $t(i) \geqq 2$. Suppose $t(i) \geqq 3$. If V_1 has the coherent (or discoherent) orientation with M_i, V_j has the discoherent (or coherent) orientation with M_i, $j=2, 3, \cdots, t(i)$. Then, V_2 and V_3 are well-situated in M_i. Since M_i is orientable, there exists no s, $1 \leqq s \leqq p$, such that V_2 and V_3 are contained in $\partial N(C_s; M)$, see Fig. 3. Therefore, $V_2 (=C_\lambda)$ and $V_3 (=C_\mu)$ are well-situated in M, $1 \leqq \lambda, \mu \leqq p$ and $\lambda \neq \mu$; which is a contradiction to the property of A. Hence $t(i)=2$.

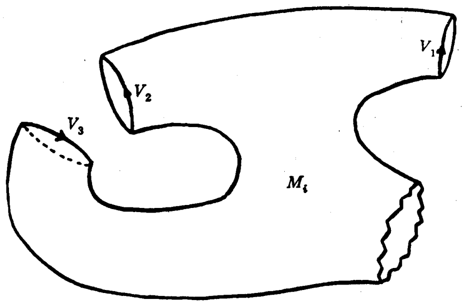


Fig. 3.

Let T_1, T_2 be components of $\partial M_i - \partial M$. If $T_1 \sim (-T_2)$ in M, T_1 and T_2 are well-situated in M which is a contradiction to the property of A, by the same reason as above. Therefore $T_1 \sim T_2$ in M, since $\partial M \sim 0$. Since M is connected, $C_1 \sim C_2 \sim C_3 \sim \cdots \sim C_p$ in M, Hence pC_1 represents a homology class θ of $H_{n-1}(M; Z)$, which is a contradiction to (2). Therefore $p=1$ and A is a connected $(n-1)$-submanifold representing θ.

[Proof of Theorem 2 (2)$\longrightarrow$(1).]

By Theorem 3 (2)$\longrightarrow$(1), a homology class

$$\frac{\theta}{|\alpha|} = \frac{a_1}{|\alpha|} g_1 + \frac{a_2}{|\alpha|} g_2 + \cdots + \frac{a_r}{|\alpha|} g_r ,$$

can be represented by a connected $(n-1)$-submanifold A in M, since $(a_1/|\alpha|, \alpha_2/|\alpha|, \cdots, \alpha_r/|\alpha|)=1$.

Let $f: A \times I \longrightarrow M$ be an embedding such that $f|_{A \times \{0\}} = id$. We have mutually disjoint $(n-1)$-submanifolds $A_i = f(A \times \{i/|\alpha|\})$, $i=1, 2, \cdots, |\alpha|$, where each A_i has the same orientation as $\alpha/|\alpha| A$'s. Let $S_1, S_2, \cdots, S_r$ be mutually disjoint $(n-1)$-spheres such that each S_i bounds n-ball B_i in M and $B_i \cap f(A \times I) = \phi$, where $\gamma = \beta - |\alpha|$. Then we have a required $(n-1)$-submanifold $A_1 \cup A_2 \cup \cdots \cup A_{|\alpha|} \cup S_1 \cup S_2 \cup \cdots \cup S_r$ in M.

To show Theorems 2 and 3 (1)$\longrightarrow$(2), it is enough to show the following;

Lemma 6. *Let M be an n-manifold without boundary or with connected boundary. If a non-trivial homology class f of $H_{n-1}(M; Z)$ is represented by a connected $(n-1)$-submanifold A in M, then αf is not representable by an $(n-1)$-submanifold having β connected components in M, where $\beta<|\alpha|$.*

Proof. Suppose αf is represented by an $(n-1)$-submanifold B having β connected components in M.

Let $g\colon A\times I\longrightarrow M$ be an embedding such that $g|_{A\times\{0\}}=id$. We have mutually disjoint $(n-1)$-submanifolds $A_i=g(A\times\{i/|\alpha|\})$, $i=1, 2, \cdots, |\alpha|$, where each A_i has the same orientation as $\alpha/|\alpha|A$'s. Since M is a manifold without boundary or with connected boundary, M is an $\tilde{L}$-manifold by Theorem 1. Since $A_1\cup A_2\cup\cdots\cup A_{|\alpha|}\sim B$, B is isotopic to an $(n-1)$-submanifold B' which is obtained from $A_1, A_2, \cdots, A_{|\alpha|}$ by a finite sequence of operations $\square_k$. While A_i and A_j are not well-situated for $i\neq j$. Hence, by Lemma 1, $A_i\square_1^{\lambda_1}\square_2^{\lambda_2}\cdots\square_{n-1}^{\lambda_{n-1}}$ and $A_j\square_1^{\mu_1}\square_2^{\mu_2}\cdots\square_{n-1}^{\mu_{n-1}}$ are not well-situated. Therefore there is no sequence of operations $\square_k$ such that $(A_1\cup A_2\cup\cdots\cup A_{|\alpha|})\square_1^{\nu_1}\cdots\square_{n-1}^{\nu_{n-1}}$ is an $(n-1)$-submanifold having β components in M, if $\beta\geqq|\alpha|$. This completes the proof.

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