THE LAW OF THE ITERATED LOGARITHM FOR THE PROCESSES GENERATED BY MIXING PROCESSES

By

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(Received February 23, 1972)

1. Summary.

Strassen [5] presented a generalization of the law of the iterated logarithm for independent random sequences and *Chover* [1] gave another proof of *Strassen's* main result. On the other hand, the authors proved in [3] and [4] that the law of the iterated logarithm and its Strassen's version hold for some classes of strictly stationary processes satisfying mixing conditions. The main object of this paper is to generalize the results in [4] to processes generated by mixing processes.

2. Preliminaries, the law of the iterated logarithm.

Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary process defined on a probability space $(\Omega, \mathfrak{B}, P)$ with $Ex_j^2<\infty$, satisfying either the uniformly strong mixing (u.s.m.) condition:

$$\sup_{A\in\mathfrak{M}_{-\infty}^{k},\, B\in\mathfrak{M}_{k+n}^{\infty}} \frac{1}{P(A)}|P(A\cap B)-P(A)P(B)|=\varphi(n)\downarrow 0\ (n\to\infty) \tag{1}$$

or the strong mixing (s. m.) condition:

$$\sup_{A\in\mathfrak{M}_{-\infty}^{k},\, B\in\mathfrak{M}_{k+n}^{\infty}} |P(A\cap B)-P(A)P(B)|=\alpha(n)\downarrow 0\ (n\to\infty) \tag{2}$$

where $\mathfrak{M}_a^b$ denotes the σ-algebra generated by the random variables x_j, $j=a, a+1, \cdots, b$. Further, let H_a^b be a Hilbert space of random variables, measurable with respect to $\mathfrak{M}_a^b$, and U an isometric operator on $H_{-\infty}^{\infty}$. Define

$$Y_j=U^jY\ (Y\in H_{-\infty}^{\infty}) \tag{3}$$

and

(4) $$S_n = Y_1 + \cdots + Y_n$$

with $S_0=0$. Put

(5) $$\sigma^2 = EY_0^2 + 2\sum_{j=1}^{\infty} EY_0 Y_j$$

if the series converges. In what follows, we assume that $\sigma^2>0$. Let

(6) $$\psi(k) = E|Y - E\{Y|\mathfrak{M}_{-k}^{k}\}|^2 .$$

The following is the results in [3].

Theorem A. *Let $\{x_j, -\infty<j<\infty\}$ be a strictly stationary process, and $\{Y_j\}$ random variables obtained by the method indicated above with $EY_j=0$. Further, let one of the following three sets of requirements be fulfilled*:

(I) (I-1) *$\{x_j\}$ satisfies the u.s.m. condition,*
(I-2) *$E|Y|^{2+\delta_1} <\infty$ for some $\delta_1>0$,*
(I-3) *$\varphi(n)=O(n^{-(1+\varepsilon)})$ for some $\varepsilon>(1+\delta_1)^{-1}$,*
(I-4) *$\psi(k)=O(k^{-(2+\delta_2)})$ for some $\delta_2>0$.*

(II) (II-1) *$\{x_j\}$ satisfies the s.m. condition,*
(II-2) *$|Y_j|<M$ with probability one,*
(II-3) *$\alpha(n)=O(n^{-(1+\delta_3)})$ for some $\delta_3>0$,*
(II-4) *$\psi(k)=O(k^{-(2+\delta_4)})$ for some $\delta_4>0$.*

(III) (III-1) *$\{x_j\}$ satisfies the s.m. condition,*
(III-2) *$E|Y|^{2+\delta_5}<\infty$ for some $\delta_5>0$;*
(III-3) *$\sum_{j=1}^{\infty} \{\alpha(j)\}^{\delta'/(2+\delta')}<\infty$ for some $0<\delta'<\delta_5$,*
(III-4) *$\psi(k)=O(k^{-(2+\delta_6)})$ for some $\delta_6>0$.*

Then, the process $\{Y_j\}$ obeys the law of the iterated logarithm.

3. Strassen's version of the law of the iterated logarithm.

Next, we consider the space C of all continuous functions on $[0, 1]$ vanishing at 0, with the usual maximum norm, and, for each $\omega\in\Omega$, define the functions $f_n(t, \omega)$, $n\geqq 3/\sigma^2$ in C as follows:

(7) $$f_n(t,\omega)=\begin{cases} S_k/\chi(n) \text{ for } t=k/n,\ k=0, 1,\cdots, n \\ \text{linearly interpolated for } t\in[k/n, (k+1)/n] \\ \quad k=0,\cdots, n-1, \end{cases}$$

where $\chi(n)=(2n\sigma^2 \log\log n\sigma^2)^{1/2}$. We denote by K the subset of C consisting of all functions $h(t)$ absolutely continuous with respect to Lebesgue measure such that $\int_0^1 \{\dot{h}(t)\}^2 dt \leqq 1$, where $\dot{h}(t)$ stands for the Radon-Nikodym derivative of h. For any integer m and any function $h \in C$, let $\Pi_m h$ be the piecewise approximation to h defined by

$$(8) \qquad (\Pi_m h)(t)=\begin{cases} h(\nu/m) \text{ for } t=\nu/m,\ \nu=0, 1,\cdots, m, \\ \text{linearly interpolated for } t\in[\nu/m, (\nu+1)/m] \\ \nu=0, \cdots, m-1. \end{cases}$$

We shall prove the following

Theorem. *Under the same assumptions in Theorem A, for almost every $\omega \in \Omega$, the sequence of functions $\{f_n(t, \omega), n\geqq 3/\sigma^2\}$ is precompact in C and its derived set is the set K.*

Proof. Firstly, we shall prove the theorem under Condition (I). Since, using the method of the proof of Theorem 6 in [3], we have

$$(9) \qquad P(\max_{1\leqq j\leqq n} |S_j|>6a\chi(n))\leqq 2P(|S_n|\geqq a\chi(n))+O((\log n)^{-3})$$

and

$$(10) \qquad \sup_{-\infty<z<\infty} |P(S_n<z\sigma\sqrt{n})-\Phi(z)|=O((\log n)^{-3})$$

(cf. (56) in [3]), where

$$\Phi(z)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-x^2/2}\, dx$$

so, using the method of the proof of Theorem 2 in [1], we have from (9) and (10) that for almost every $\omega\in\Omega$ the sequence of functions $\{f_n(t, \omega), n\geqq 3/\sigma^2\}$ is equicontinuous.

Now, we shall prove that for almost all $\omega\in\Omega$, the derived set of the sequence of functions is contained in K. To prove this, it suffices to show (see [1] and [4]) that

$$(11) \qquad \sum_r P(A_r)<\infty$$

where

$$A_r=\left\{\omega \,\middle|\, m\sum_{\nu=0}^{m-1}\left[(\Pi_m f_{n_r})\left(\frac{\nu+1}{m}, \omega\right)-(\Pi_m f_{n_r})\left(\frac{\nu}{m}, \omega\right)\right]^2>(1+\varepsilon)^2\right\}$$

and $n_r=[c^r]$ with some suitably chosen $c=c(\varepsilon)>1$. Let i be the smallest integer such that $i/n_r\geqq\nu/m$ and j the largest integer such that $j/n_r<(\nu+1)/m$. Let

$$\xi_{r,\nu}=(2m\log\log n_r\sigma^2)^{1/2}\left\{(\Pi_m f_{n_r})\left(\frac{\nu+1}{m}\right)-(\Pi_m f_{n_r})\left(\frac{\nu}{m}\right)\right\}$$

(12)
$$=(2m\log\log n_r\sigma^2)^{1/2}\{1/\chi(n_r)\sum_{k=i}^{j}Y_k+y_{r,\nu}\}$$

$$=\frac{1}{\sigma\sqrt{n_r/m}}\sum_{k=i}^{j}Y_k+(2m\log\log n_r\sigma^2)^{1/2}y_{r,\nu}\quad \nu=0,1,\cdots,m-1\ ,$$

where

$$y_{r,\nu}=(\Pi_m f_{n_r})\left(\frac{i}{n_r},\omega\right)-(\Pi_m f_{n_r})\left(\frac{\nu}{m},\omega\right)+(\Pi_m f_{n_r})\left(\frac{\nu+1}{m},\omega\right)-(\Pi_m f_{n_r})\left(\frac{j}{n_r},\omega\right).$$

Furthermore, let $N_{r,\nu}$ denote the number of summands of the first term, $j-i$, which is $\sim n_r/m$. Put $q_r=[N_{r,\nu}^{(1-\beta)/2}]$, with some $0<\beta<\delta_1/(2+\delta_1)$, and let

(13)
$$\eta_{r,\nu}=\frac{1}{(N_{r,\nu}-2q_r)^{1/2}\sigma}\sum_{k=i+q_r+1}^{j-q_r}Z_k^{(q_r)},\quad \nu=0,1,\cdots,m-1$$

where

(14)
$$Z_k^{(s)}=E\{Y_k|\mathfrak{M}_{k-s}^{k+s}\}\ .$$

It follows from the proof of Theorem 18. 6. 1 in [2], that $\{Z_j^{(s)}\}$ is a strictly stationary process with $EZ_j^{(s)}=0$ and satisfies the u.s.m. condition with the function $\varphi_Z(n)$ for which

(15)
$$\varphi_Z(n)\leqq\begin{cases}1, & n\leqq 2s\\ \varphi(n-2s), & n>2s\end{cases}$$

and

(16)
$$\sup_{A\in\mathfrak{M}_{-\infty}^{k}(Z),B\in\mathfrak{M}_{k+n}^{\infty}(Z)}\frac{1}{P(A)}|P(A\cap B)-P(A)P(B)|\leqq\varphi_Z(n)$$

where $\mathfrak{M}_a^b(Z)$ is the σ-algebra generated by the $Z_j^{(s)}$, $j=a, a+1, \cdots, b$. Let

$$\bar{Z}_j^{(s)}=Y_j-Z_j^{(s)}$$

and define $\zeta_{r,\nu}(\nu=0,1,\cdots, m-1)$ as

$$\begin{aligned}\zeta_{r,\nu}&=\xi_{r,\nu}-\eta_{r,\nu}\\&=\frac{1}{(N_{r,\nu}-2q_r)^{1/2}\sigma}\{\sum_{k=i}^{i+q_r}Y_k+\sum_{k=j-q_r+1}^{j}Y_k+\sum_{k=i+q_r+1}^{j-q_r}\bar{Z}_k^{(q_r)}\}\\&\quad+(2m\log\log n_r\sigma^2)^{1/2}y_{r,\nu}\ .\end{aligned}$$

Then

$$\begin{aligned}E|\zeta_{r,\nu}|^2&=\frac{1}{(N_{r,\nu}-2q_r)\sigma^2}E|\sum_{k=i+1}^{i+q_r}Y_k+\sum_{k=j-q_r+1}^{j}Y_k+\sum_{k=i+q_r+1}^{j-q_r}\bar{Z}_k^{(q_r)}|^2\\&\quad+\frac{2(2m\log\log n_r\sigma^2)^{1/2}}{(N_{r,\nu}-2q_r)^{1/2}\sigma}E\{(\sum_{k=i+1}^{i+q_r}Y_k+\sum_{k=j-q_r+1}^{j}Y_k+\sum_{k=i+q_r+1}^{j-q_r}\bar{Z}_k^{(q_r)})y_{r,\nu}\}\\&\quad+2m\log\log n_r\sigma^2E|y_{r,\nu}|^2\ .\end{aligned}$$

Since

$$E|\sum_{k=1}^{t}Y_k|^2\leqq t^2E|Y|^2$$

and

$$E|\sum_{k=i+q_r+1}^{j-q_r}\bar{Z}_k^{(q_r)}|^2\leqq(j-i-2q_r)^2E|\bar{Z}_k^{(q_r)}|^2=(N_{r,\nu}-2q_r)^2\psi(q_r)$$

so, from (I-4) and the definition of q_r that there exists a positive number γ such that

$$E|\zeta_{r,\nu}|^2=O(n_r^{-\gamma})$$

and hence

$$\begin{aligned}&E|\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2-\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2|\leqq2\sum_{\nu=0}^{m-1}E|\eta_{r,\nu}\zeta_{r,\nu}|+\sum_{\nu=0}^{m-1}E|\zeta_{r,\nu}|^2\\&\leqq2\sum_{\nu=0}^{m-1}\{E|\eta_{r,\nu}|^2\}^{1/2}\{E|\zeta_{r,\nu}|^2\}^{1/2}+\sum_{\nu=0}^{m-1}E|\zeta_{r,\nu}|^2=O(n_r^{-\gamma/2})\ .\end{aligned}$$

Therefore, by *Chebyshev's* inequality, we have, for sufficiently large r,

$$\begin{aligned}P(A_r)&=P(\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2>(1+\varepsilon)^2(2\log\log n_r\sigma^2))\\&\leqq P(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2>(1+\varepsilon)^2(2\log\log n_r\sigma^2)-n_r^{-\gamma/4})+P(|\sum_{\nu=0}^{m-1}\xi_{r,\nu}^2-\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2|\geqq n_r^{-\gamma/4})\\&\leqq P(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2>(1+\varepsilon')(2\log\log n_r\sigma^2))+O(n_r^{-\gamma/4})\end{aligned}$$

where $\varepsilon'>0$ with $1+\varepsilon'<(1+\varepsilon)^2$. As $\{Z_j^{(q_r)}\}$ satisfies Condition (II) of Theorem 1 in [4] with the function $\varphi_Z(n)$, so from Theorem 3 in [4]

$$\sum_r P(\sum_{\nu=0}^{m-1}\eta_{r,\nu}^2>(1+\varepsilon')(2\log\log n_r\sigma^2))<\infty$$

which implies (11).

Finally, using (10), we can prove that K is contained in the derived set of $\{f_n(t,\omega),\ n\geqq 3/\sigma^2\}$ (cf. The proofs of Lemma 5 in [1] and Theorem 5 in [4]).

Thus, we complete the proof of the theorem under (I). The proof under either (II) or (III) is carried out by the same method as above. Thus, the proof is completed.

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