EXTENSIONS OF VECTOR MEASURES

By

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1. Introduction.

A fundamental problem in measure theory is that of finding conditions under which countably additive measure on a ring or a field R can be extended to a countably additive measure on a wider class of sets containing R. In case m is a non-negative measure, this problem is essentially solved by the Carathéodory process of generating an outer measure m^* and taking the family of m^*-measurable sets (see *Halmos* [9] § 10—§ 13). In recent years many authors have considered the extension ploblem when the range of m is contained in a vector space X or a topological group X. If X is a Banach space, then there are solutions due to *Găină* [7] (in the case m has finite variation), *Kluvanek* [11], [12] *Dinculeanu and Kluvanek* [4] (in the case m is absolutely continuous with respect to a non-negative measure), *Gould* [8] (in the case m is locally bounded and X satisfies axiom (A), he has generalized the notions of outer measure and measurable sets). If X is a commutative complete topological group and m is of bounded variation, then there is a very nice extension theorem by *Takahashi* [16].

In this paper, we shall obtain another condition (Theorem 1 and Theorem 3 (3)) which is equivalent to other ones already considered. Now we shall consider the extension theorem under which X is a special Banach space.

2. Extensions of vector measures.

Let S be a set and Σ a field (algebra) of subsets of S. Then there exists the smallest σ-field $\sigma(\Sigma)$ containing Σ. The σ-field $\sigma(\Sigma)$ is called the σ-field generated by Σ.

Let X be a Banach space and X^* its dual.

Definition 1. *A set function m defined on Σ with values in X is called a vector measure if for every sequence $\{E_n\}$ of mutually disjoint sets of Σ with $E=\bigcup_{n=1}^{\infty} E_n \in \Sigma$ we have $m(E)=\sum_{n=1}^{\infty} m(E_n)$.*

For every set $E\subset S$ we put $\tilde{m}(E)=\sup\{\|m(A)\| : A\subset E,\ A\in\Sigma\}$. Then $\tilde{m}(E)$ has the following properties:

(i) $0\leqq\tilde{m}(E)\leqq+\infty$.

(ii) $E_1\subset E_2 \Longrightarrow \tilde{m}(E_1)\leqq\tilde{m}(E_2)$.

(iii) $\tilde{m}(E_1\cup E_2)\leqq\tilde{m}(E_1)+\tilde{m}(E_2)$ for every sets $E_1, E_2\in\Sigma$.

Theorem 1. *Every vector measure* m: $\Sigma\to X$ *can be extended to uniquely a vector measure* m_1: $\sigma(\Sigma)\to X$ *if and only if*
(0) for every sequence $\{E_n\}$ *of mutually disjoint sets of* Σ *we have* $\lim_{n\to\infty}\|m(E_n)\|=0$.

Proof. The necessity is obvious.
Sufficiency. By *Brooks* ([18] Theorem 2) there exists a positive bounded measure ν on Σ such that $\lim_{\nu(A)\to 0}\|m(A)\|=0$ and $\nu(E)\leqq\tilde{m}(E)$ $(E\in\Sigma)$. By *Halmos* ([9] § 13. Theorem A) ν has a unique extension $\bar{\nu}$ on $\sigma(\Sigma)$. The boundedness of $\bar{\nu}$ is obvious. We put $\rho(E_1, E_2)=\bar{\nu}(E_1\Delta E_2)=\bar{\nu}(E_1-E_2)+\bar{\nu}(E_2-E_1)$ for every sets $E_1, E_2\in\sigma(\Sigma)$. Then we can consider on $\sigma(\Sigma)$ the uniform structure τ defined by the semi-distance ρ and by *Halmos* ([9] § 13. Theorem D) $\Sigma\subset\sigma(\Sigma)$ is dense in $\sigma(\Sigma)$ for the topology induced by τ. Since $\lim_{\bar{\nu}(A)\to 0}\|m(A)\|=\lim_{\nu(A)\to 0}\|m(A)\|=0$, $A\in\Sigma$, by *Dinculeanu* and *Kluvanek* ([4] Theorem 2) m can be extended to a vector measure m_1: $\sigma(\Sigma)\to X$ such that $\lim_{\bar{\nu}(A)\to 0}\|m_1(A)\|=0$, $A\in\sigma(\Sigma)$.
The uniqueness of m_1 is immediate by *Dinculeanu* ([2] § 2. Proposition 6).

Corollary. *The condition (0) is equivalent to*
(0′) for every sequence $\{E_n\}$ *of mutually disjoint sets of* Σ *the series* $\sum_{n=1}^{\infty} m(E_n)$ *converges unconditionally.*

Proof. $(0')\Longrightarrow(0)$. It is obvious.
$(0)\Longrightarrow(0')$. By Theorem 1 m has an unique extension m_1: $\sigma(\Sigma)\to X$. For every sequence $\{E_n\}$ of mutually disjoint sets of Σ we have $m_1(\bigcup_{n=1}^{\infty} E_n)=\sum_{n=1}^{\infty} m(E_n)$ and $\sum_{n=1}^{\infty} m(E_n)$ converges unconditionally.

Example 1. Every finite non-negative measure on Σ satisfies the condition (0) of Theorem 1.

Example 2. If m has finite variation, then m satisfies the condition (0) of Theorem 1.

Example 3. (*Dinculeanu and Kluvanek* [4]) Let S be an uncountable infinite set, Σ the field consisting of the finite subsets of S and their complements and X the Banach space of bounded function on S with the sup-norm. Define m: $\Sigma \to X$ by $m(A) = \varphi_A$ if A is finite and $m(A) = -\varphi_{S-A}$ if $S-A$ is finite (where φ_A is the characteristic function of set A). Then m is a vector measure. If $S-A$ is finite, then there exists a sequence $\{a_n\}$ of points of A. Since $\|m(\{a_n\})\| = \|\varphi_{a_n}\| = 1$ $(n=1, 2, \cdots)$, m not satisfies the condition (0).

Theorem 2. *Let R be a ring and $\sigma(R)$ the σ-ring generated by R. Every vector measure m: $R \to X$ can be extended to uniquely m_1: $\sigma(R) \to X$ if and only if one of the following conditions is satisfied.*

(1) *there exists a positive bounded measure ν on R such that $\lim_{\nu(A)\to 0}\|m(A)\|=0$, $A \in R$ (Dinculeanu and Kluvanek* [4] *Theorem 2, Corollary 1).*

(2) *m satisfies the condition (0) of Theorem 1.*

(3) *the set $\{m(E): E \in R\}$ is conditionally weakly compact (Kluvanek* [12] *Theorem* 4.1).

Proof. The necessity of (1) is obvious.

(1)$\Longrightarrow$(2). Since ν satisfies the condition (0) and $\lim_{\nu(A)\to 0}\|m(A)\|=0$, it is obvious.

(2)$\Longrightarrow$(3). It is obvious from *Brooks* ([18] Theorem 1, Corollary). The sufficiency of (3) is obvious from *Kluvanek* ([12] Theorem 4.1).

3. Vector measures on a ring

Definition 2. A non-void class φ of subsets of S is called a δ-ring if

(1) $A, B \in \varphi \Longrightarrow A \cup B \in \varphi$, $A - B \in \varphi$.

(2) $A_n \in \varphi (n=1, 2, \cdots) \Longrightarrow \bigcap_{n=1}^{\infty} A_n \in \varphi$.

Let R be a ring of subsets of S. Then there exists the smallest δ-ring $\varphi(R)$ containing R (*Dinculeanu* [2] §1. Proposition 6). The δ-ring $\varphi(R)$ is called the δ-ring generated by R.

Let X be a Banach space and X^* its dual.

Theorem 3. *Every vector measure m: $R \to X$ can be extended to uniquely a vector measure m_1: $\varphi(R) \to X$ if and only if one of the following conditions is satisfied.*

(1) *for every set $E \in R$ there exists a finite non-negative measure ν_E on R such that $\lim_{\nu_E(A)\to 0}\|m(A)\| = 0$, $A \subset E$, $A \in R$ (Dinculeanu and Kluvanek* [4] *Theorem 2, Corollary 2).*

(2) *for every set* $E\varepsilon R$ *and every number* $\varepsilon>0$ *there exists a positive integer* n *such that if* $E_i\in R$, $E_i\subset E$ $(i=1, 2, \cdots, n)$ *and* $E_i\cap E_j=\phi(i\neq j)$, *then there exists a positive integer* i_0 $(1\leqq i_0\leqq n)$ *such that* $\|m(E_{i_0})\|<\varepsilon$ (*Takahashi* [16] *Theorem 1*).
(3) *for every set* $E\in R$ *and every sequence* $\{E_n\}$ *of mutually disjoint sets of* R *with* $E_n\subset E$ $(n=1, 2, \cdots)$ *we have* $\lim_{n\to\infty}\|m(E_n)\|=0$.

(4) *for every set* $E\in R$ *the set* $\{m(F)\colon F\subset E, F\in R\}$ *is conditionally weakly compact in* X (*Kluvanek* [13] *Theorem 5.3*).

Proof. By *Dinculeanu* and *Kluvanek* ([4] Theorem 2, Corollary 2) (1) is the necessary (and sufficient) condition in order that m has a countably additive extension m_1: $\varphi(R)\to X$.
(1)$\Longrightarrow$(2). Since ν_E is finite, ν_E satisfies the condition (2). Then m satisfies the condition (2), since $\lim_{\nu_E(A)\to 0}\|m(A)\|=0$, $A\subset E$, $A\in R$.
(2)$\Longrightarrow$(3). It is obvious.
(3)$\Longrightarrow$(4). For every set $E\in R$ we put $\Sigma(E)=\{F\colon F\subset E, F\in R\}$.
Then $\Sigma(E)$ is a field of subsets of E and m satisfies the condition (0) of Theorem 1. Therefore m can be extended to uniquely a vector measure m_1: $\sigma(\Sigma(E))\to X$. By *Bartle, Dunford* and *Schwartz* ([1] Theorem 2.9) the set $\{m_1(A)\colon A\in\sigma(\Sigma(E))\}$ is conditionally weakly compact and hence the set $\{m(A)\colon A\in\Sigma(E)\}$ so is.
The sufficiency of (4) is immediate by *Kluvanek* ([13] Theorem 5.3).

Corollary. *The following is equivalent to the condition* (3) *of Theorem 3,*
(3′) *for every set* $E\in R$ *and every sequence* $\{E_n\}$ *of mutually disjoint sets of* R *with* $E_n\subset E$ $(n=1, 2, \cdots)$ *the series* $\sum_{n=1}^{\infty}m(E_n)$ *converges unconditionally.*

The proof is obvious.
We shall now consider the extension theorem in the case X is a special Banach space.

Theorem 4. *Let* X *be a Banach space such that*
(*A*) *if* $\{x_n\}$ *is a sequence in* X *whose norms have a positive lower bound, then there exists for arbitrary positive number* K *a finite subsequence* $\{x_{n_r}\}$ *such that* $\|\sum_r x_{n_r}\|>K$.
Then m: $R\to X$ *has a countably additive extension* m_1: $\varphi(R)\to X$ *if and only if* m *is locally bounded over* R, *that is, for every set* $E\in R$ $\tilde{m}(E)<+\infty$ (*Gould* [8] § 4).

Proof. Necessity. Since for each $x^*\in X^*$ x^*m_1 is a scalar measure of the δ-ring $\varphi(R)$, by *Dinculeanu* ([2] § 3, Proposition 14) we have $\widetilde{x^*m_1}(E)=\sup\{|x^*m_1(A)|$:

$A\subset E,\ A\in\varphi(R)\}<+\infty$ for every set $E\in\varphi(R)$. By uniform boundedness theorem $\tilde{m}_1(E)<+\infty$. Therefore $\tilde{m}(E)\leqq\tilde{m}_1(E)<+\infty$ for every set $E\in R$.
Sufficiency. We shall show that the locally boundedness of m implies (3) of Theorem 3. If it is false, then there exist a set $E\in R$, a number $\varepsilon>0$ and a sequence $\{E_n\}$ of mutually disjoint sets of R with $E_n\subset E$ $(n=1, 2, \cdots)$ such that $\|m(E_n)\|>\varepsilon$ for all n. By hypothesis (A) for any positive number K there exists a finite subsequence $\{m(E_{n_r})\}$ such that $\|\sum_r m(E_{n_r})\|>K$. Since $\|\sum_r m(E_{n_r})\|$ $=\|m(\bigcup_r E_{n_r})\|>K$, we have $\tilde{m}(E)=+\infty$. Therefore we have a contradiction.

Corollary 1. *Let X be a weakly complete Banach space. Then m has a countably additive extension m_1: $\varphi(R)\to X$ if and only if for every $x^*\in X^*$ the scalar measure x^*m has finite variation $\bar{x}^*\bar{m}$.*

Proof. Necessity. By *Gould* ([8] Theorem 3.1) X satisfies the hypothesis (A) in Theorem 4. By *Dinculeanu* ([2] § 3. Proposition 7) $\tilde{x}^*\tilde{m}(E)\leqq\bar{x}^*\bar{m}(E)\leqq4\cdot\tilde{x}^*\tilde{m}(E)$ $\leqq4\cdot\|x^*\|\cdot\tilde{m}(E)$. Since $\tilde{m}(E)<+\infty$ from the above theorem, we have $\bar{x}^*\bar{m}(E)<+\infty$.
Sufficiency. It is obvious from the inequality $|x^*m(E)|\leqq\bar{x}^*\bar{m}(E)$ and the uniform boundedness theorem.

Corollary 2. *If X is a reflexive Banach space and sup $\{\|m(E)\|:E\in R\}<+\infty$, then m has a countably additive extension m_1: $\varphi(R)\to X$ (Fox [5] Theorem).*

Proof. Since X is reflexive, X is weakly complete. It is obvious, since $\bar{x}^*\bar{m}(A)\leqq4\cdot\|x^*\|\cdot\sup\{\|m(E)\|:\ E\in R\}<+\infty$.

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