LOCALLY UNKNOTTED SETS IN THREE-SPACE

By

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1. Codimension 2.

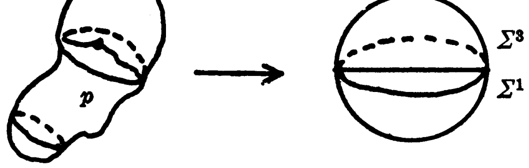
A polyhedral (or tame) simple closed curve is called unknotted if it is equivalent to a plane curve under a autohomeomorphism of the underlying space (ambient homeomorphism). Such a curve is the boundary of a non-singular disk. Conversely, if a polyhedral curve bounds a disk it is known to be unknotted [13]. By the *Dehn-Papakyriakopoulus* theorem a curve is unknotted iff the corresponding knot group is an infinite cyclic group. Hence in this situation we have

boundary of a disk=unknotted=abelian infinite cyclic knot group.

Wild curves and arcs were discovered by Antoine and dealt with decisively by *Fox-Artin* [8]. In order to motivate the following definitions we recall specifically the examples 1.2 and 1.4 of that paper as well as the remarkable simple closed curve of *R. H. Fox* [7]. [See figures 2, 3 and 4, respectively, of the displayed examples below.]

Let M be a 1-manifold in E^3. Let Σ^3 be a standard 3-ball, $\Sigma^k=\{(x_1, \cdots, x_k)| \Sigma x_i^2\leq 1\}$, p an arbitrary point of M. Then M is called *locally flat* at p if there exists a 3-ball B whose interior contains p and a homeomorphism h of the pair $(B, B\cap M)$ onto the pair (Σ^3, Σ^1).

LF



$h(B, B\cap M)\rightarrow(\Sigma^3, \Sigma^1)$

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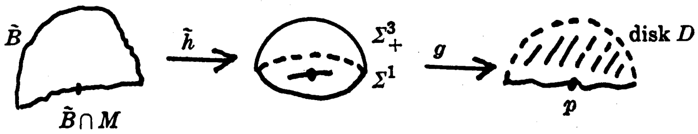
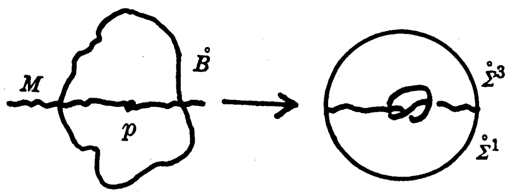
If the top half of the standard 3-ball is denoted by Σ^3_+, etc., the existence of the B, h above imply (by restriction) the existence of a pair $\tilde{B}$, $\tilde{h}$ such that

LU 

$$\tilde{h}(\tilde{B}, \tilde{B}\cap M) \to (\Sigma^3_+, \Sigma^1)$$

If at a given point p of M, a pair $\tilde{B}$, $\tilde{h}$ exists as above we call M *locally unknotted* at p. (Example 1.4 does *not* have this property at all points.) The map g may be regarded as a pseudo-isotopy of Σ^3_+ onto a disk D. By anology with the polyhedral case, we now say M is *locally* unknotted at p.

Again, if M is LF at p, and $\varepsilon>0$, there is a topological sphere $\mathring{B}$ of diameter $<\varepsilon$ whose interior contains p and a homeomorphism of the pair $(\mathring{B}, B\mathring{\cap} M)$ onto the pair $(\mathring{\Sigma}^3, \mathring{\Sigma}^1)$

LPU 

$$h(\mathring{B}, B\mathring{\cap} M) \to (\mathring{\Sigma}^3, \mathring{\Sigma}^1)$$

We refer to this (strictly local) property as *local peripheral unknottedness.* Example 1.2 does *not* have this property at *all* points. These examples show that LU and LPU are independent in condimension 2. If these properties hold at all points [13]

$$\mathrm{LU}+\mathrm{LPU}=\mathrm{LF}\,.$$

A simple closed curve k is called *almost unknotted* if it contains a point p such that if $\varepsilon>0$, there is a positive δ and a homeomorphism $h\colon E^3 \twoheadrightarrow E^3$ such $h=id$ on $S(p, \delta)$ and $h|k\backslash S(p, \varepsilon)$ is a subset of a plane. There is a wild almost unknotted curve (figure 4).

It turns out that if k is the "Remarkable..., then k fails to be either SLU (see definition page 54) or LPU at the point p.

In view of characterizations of the class of tame arcs (and simple closed curves modulo the knot problem) it is natural to ask how many *other* equivalence classes there are and to give a characterization of one or more of them. That there are uncountably many such classes was observed by *Alford-Ball* [2] and by *Fox-Harrold* [9].

As mentioned above the example 1.4 has one point at which it is locally knotted. It is a union two tame arcs having an intersection a common end-point. A special class of these so-called mildly wild arcs, namely, the Wilder arcs have been catalogued. A Wilder arc is a mildly wild arc that is LPU at each point. An example of *Lomonaco* shows that mildly wild, non-Wilder arcs exist [16] (figure 5).

Perhaps the next simplest class of arcs whose embedding type might be analyzed is that such that each representative is LU at all points but fails to be LPU at one point. If the exceptional point is an interior point of the arc, the phenomena discovered by *Lomonaco* will have to be reckoned with.

The generalization of LPU leads to the penetration index of *Alford and Ball*. The related concepts of enveloping genus due to *R. H. Fox* and local enveloping genus due to *R. B. Sher* lead to some unexpected phenomena [20], which we will now describe:

Some auxiliary notions

The penetration index of an arc at a point x

$$P(K, x)=\lim_{\varepsilon\to 0}\inf_{\text{all } U} \text{card}\, K\cap\{\bar{U}(x, \varepsilon)\setminus U(x, \varepsilon)\} .$$

The penetration index of an arc or simple closed curve K equals $\text{lub}_x P(K, x)$. If K has a single wild point q $P(K)=P(K, q)$.

The enveloping genus of a 1-dim continuum A (Fox), EG(A) is defined as follows: Suppose

$$A=\bigcap_{1}^{\infty} M_i$$

where $A\subset \text{int}\, M_i\subset M_i\subset \text{int}\, M_{i-1}$, and $M_1, M_2, \cdots$ are bounded 3-manifolds. Then

$$\text{EG}(A)=\text{smallest cardinal } m$$

such that for some sequence $M_1, M_2, \cdots$

$$\text{genus Bd}\, M_i\leq \text{m} .$$

If no such m exists, $\text{EG}(A)=\aleph_0$.

The local enveloping genus of a simple closed curve A at p is the smallest cardinal m such that there exists arbitrarily small closed, connected surfaces of genus m whose interior contains p that meet A in exactly two points.

Sher shows that for a closed curve K with exactly one bad (=wild) point [20]

$$\mathrm{EG}(K)=1+\mathrm{LEG}(K)$$

and

$$2\mathrm{EG}(K)\leq P(K)\,.$$

If $P(K)\geqq 6$, we may have $2\mathrm{EG}(K)<P(K)$.

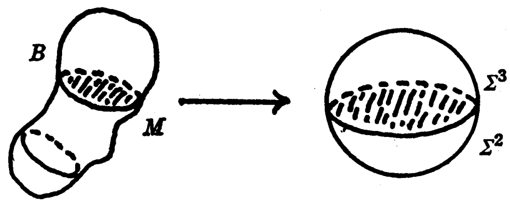
2. Codimension 1.

The first wild surfaces were described by *Antoine* and somewhat later by *Alexander*. Although the *Jordan-Brouwer* theorem in higher dimensions had been established some ten years earlier, these examples were significant in showing that a Schoenflies type proof of the Jordan separation theorem, $n>2$, cannot be given.

The definition of locally flat surface M at a point p is analogous to the definition in codimension 2. We say M is LF at p iff there is a ball B whose interior contains p and a homeomorphism h such that

$$h:(B,\, B\cap M)\rightarrow(\Sigma^3,\, \Sigma^2)$$

LF



If $B\cap M$ divides B into B_+ and B_- the existence of h, B as above implies that if $h_+=h|B_+$, $h_-=h|B_-$, then

$$h_+:(B_+,\, B_+\cap M)\rightarrow(\Sigma^3_+,\, \Sigma_2)\,,$$
$$h_-:(B_-,\, B_-\cap M)\rightarrow(\Sigma^3_-,\, \Sigma_2)\,.$$

The surface is called *locally unknotted* at p if such a B and h exists. Clearly, the existence of h_+, B_+ and h_-, B_- together that agree on $B\cap M$ implies such a B and h exist. In codimension 1, LU and LF are the same.

All of the *Fox-Artin* examples of wild arcs (except 1.4) can be "blown up" to yield wild surfaces with one or more bad points. The "Remarkable simple closed curve" cannot be blown up to lie on a solid torus as we shall see a little later.

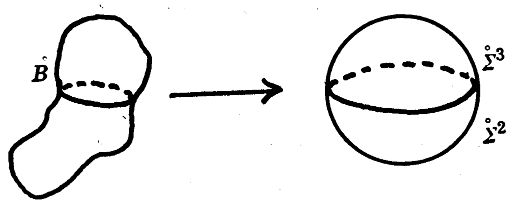
In 1961 *R. H. Bing* [4] gave an essentially new example of a wild surface based on his cube-with-eye-bolt construction. Since this example has been quite fruitful, we refer to fig. 7 which illustrates a typical step in the construction. The surface described by *Bing* is wild but every arc or simple closed curve on it is tame. In *Bing*'s example one of the complementary domains is not simply connected. By modifying the construction slightly *Gillman* also produced such an example so that both complementary domains are simply connected.

A weaker condition than LU

The surface M is called *locally peripherally unknotted* at p if for each $\varepsilon>0$ there is a topological sphere $\mathring{B}$ of diameter less than ε enclosing p and a homeomorphism

$$h: (\mathring{B}, B \mathring{\cap} M) \to (\mathring{\Sigma}^3, \mathring{\Sigma}^2)$$

LPU



Clearly LU $\Rightarrow$ LPU. That the converse does not hold is easily seen by the following example.

Let C be a topological disk in E^3 obtained by swelling up the arc of 1.2 to a disk so that the disk is locally tame except at one interior point of C. Let D be a planar disk from which a sequence of disjoint open 2-cells $D_1, D_2 \cdots$ have been removed. Suppose $\lim D_i = p$ an interior point of the rectangle. Now sew into each hole D_i a copy of C, C_i with the diameter of $C_i \to 0$ with $1/i$. Then the so modified C is a disk that is LPU at p, but the modified C is not tame. *If*, however, the surface is LPU at *all* points then LPU=LU [*Harrold*, Annals of Math. 69 (1959), 276–290].

An account of recent results that have been attained by weakening the LPU condition by using "locally spanning disks" and related ideas may be found in a recent paper by *Burgess and Cannon* in the Rocky Mountain Journal of Mathematics [6].

A 2-sphere S in E^3 is called *locally spanned* from the component U of $E^3 \backslash S$ at p iff $\forall \varepsilon>0$, $\exists \varepsilon$-disks D and $D' \ni p \in \operatorname{int} D \subset S$, $\operatorname{Int} D' \subset U$ and $\operatorname{Bd} D = \operatorname{Bd} D'$.

Theorem (*Burgess*). *A 2-sphere S in E^3 is tame from U iff it can be spanned from U for all $p \in S$* [5].

The following question seems unanswered at the time *Burgess* and *Cannon*'s paper went to press.

Is a 2-sphere S in E^3 tame if $\forall p \in S$, $\forall \varepsilon>0$, there is a 2-sphere S' of diam $<\varepsilon$ such that $p \in \operatorname{int} S'$ and $S \cap S'$ is a continuum?

Eaton has solved a special case, namely, when $S \cap S'$ irreducibly separates S.

Alford outlined a procedure for modifying the construction of *Bing* mentioned previously to find a surface S and a simple closed curve J on S such that S is locally tame modulo J and J pierces no disk. This implies that J fails to be LPU at any point. Clearly J is LU at each point. An example had been given earlier of a simple closed curve that fails to be either LU or LPU at any point (the so-called *Bing-sling* [3]) see figure 8.

The question of whether or not the hypothesis of LU imposed on the embedding of a 1-manifold in three space implies that the 1-manifold is a subset of a surface has been considered only recently.

3. Codimension 2 revisited.

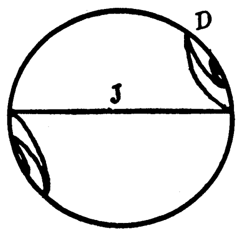
Some questions about locally unknotted curves.

In the original definition of LU for 1-manifolds it was required that at the point p of the curve J that there is a disk D whose boundary contains a neighborhood of p in J [11]. Several authors have later required only that a neighborhood of p in J lie on a disk. That the first condition is stronger than the second is clear. In the converse direction we consider three cases.

If $p \in \operatorname{int} D$, we may without loss of generality, suppose $J \subset \operatorname{int} D$. Then J does not separate D and well known accessibility properties of $D \backslash J$ at the endpoints of J allow us to form a simple closed curve K lying in the interior of D, $K \supset J$ and there is a disk D^* on whose boundary J lies.

If $p \in \operatorname{Bd} D \cap \operatorname{int} J$ we proceed as follows. The set $D \backslash p$ is connected and contains $J \backslash p = U \cup V$, where U and V are components of $J \backslash p$. Let Γ be an arc in $D \backslash p$ joining a point of U and a point of V. It may be assumed no sub-arc of Γ has this property. Then Γ plus a sub-arc of J containing p forms a simple closed curve that bounds a disk D^* of D and a neighborhood of p in J lies on the boundary of D^*.

It will be noted that in the above cases D^* is a subset of the given disk D. In the remaining case, *i.e.*, $p \in \operatorname{Bd} D \cap \operatorname{Bd} J$ this seems no longer possible, as the following diagram illustrates.



If the disk D is a subset of no other disk in E^3 and if the boundary of D pierces no disk it is not clear that a disk D^* exists whose boundary contains J.

Recently Professor *L. V. Keldyš* proved every LU 1-manifold is a subset of a 2-manifold with boundary [14].

One result of her constructions is that if an arc is LU, then it is a subset of some disk in three-space. Precisely, she has proved that for every LU arc l in E^3 there is a pseudoisotopy $F_t: E_3 \twoheadrightarrow E_3$ of the identity map, carrying a straight line segment I to l such that $F_1|I$ is a homeomorphic mapping of I on l and the set of points $x \in E^3$ for which the sets $F^{-1}(x)$ are non-degenerate is 0-dimensional and is contained in the set of wild points of l.

An example of *C. D. Sikkema* shows that the LU hypothesis cannot be omitted and obtain such a conclusion [21].

Professor *Keldyš* result shows that the part of a 1-manifold that lies on a 2-manifold can be "lengthened," provided the 1-manifold is LU.

The starting point of the present investigation was to find conditions such that the part of a 1-manifold that lies on the boundary of a 2-manifold can be "lengthened."

Our original conjecture was that a LU simple closed curve k is a boundary component of some annulus. We verify this below for *strongly* locally unknotted curves. However, whether every SLU curve (=strongly locally unknotted curve) bounds an orientable surface remains open.

Strongly locally unknotted curves.

Let x be a point of a connected 1-manifold A. We say A is locally unknotted at x if and only if there exists a disk D whose boundary, ∂D, contains a neighborhood of x in A.

By the approximation theorem of *Bing*, it may be assumed without loss of generality that D is locally polyhedral except at the points of A. If A is locally unknotted at each point we say A if locally unknotted and abbreviate by saying A is LU. If A is LU and x is a fixed point of A we note by $\mathscr{D}_x$ the collection of all disks that are locally polyhedral mod A such that $A \cap \partial D$ is connected. Similarly $\mathscr{D}_y$ denotes the collection of all disks whose boundaries contain a neighborhood of y in A satisfying similar conditions.

We say that the families $\mathscr{D}_x$ and $\mathscr{D}_y$ are contiguous provided the following is true:

Given $D_x \in \mathscr{D}_x$, there exists a $D_y \in \mathscr{D}_y$ such that $D_x \cup D_y$ contains an irreducible sub-arc of A from x to y, and, according as this arc is unique or not

we assume case α and case β as follows:
In case α it is assumed

$(D_x \cap D_y)\backslash A$ consists of a topological ray r such that $\bar{r}\backslash r=q$ is a point of A and $r \cap \partial D_x = r \cap \partial D_y$ is the unique end-point of r.

In case β it is assumed

$(D_x \cap D_y)\backslash A$ consists of a pair of disjoint topological rays r and r_1 such that $\bar{r}\backslash r=q$ is a point of A and $r \cap \partial D_x = r \cap \partial D_y$ is the unique end-point of r, $\bar{r}_1\backslash r_1=q_1$ is a point of A and $r_1 \cap \partial D_x = r_1 \cap \partial D_y$ is the unique end-point of r_1.

The 1-manifold A without boundary is called *strongly locally unknotted* if it is true that when x and y are sufficiently near one another, the families $\mathscr{D}_x$ and $\mathscr{D}_y$ are contiguous families.

A 1-manifold A with boundary is called strongly locally unknotted iff its interior is strongly locally unknotted.

Theorem α. *If the arc A is strongly locally unknotted and if A is locally unknotted at its end-points there is a disk E such that $\partial E \supset A$.*

Theorem β. *If the simple closed curve J is strongly locally unknotted, there is an annulus R such that J is a boundary component of R.*

Proof of Theorem α. Let A be parameterized so that $u=0$ determines an end-point a of A and $u=1$ determines the end-point b. Define $X=\{u=0\} \cup \{$all points $t \in A$ such that the sub-arc $0 \leq u \leq t$ lies on the boundary of some disk $E_t\}$. By the definition of LU, it follows that $X \neq \square$. To see that X is open, let H be a disk whose boundary, ∂H, contain $[0, t]$ as a subset. Since A is strongly locally unknotted there is a $t' < t$ such that $\mathscr{D}_{t'}$ and $\mathscr{D}_t$ are contiguous. That is, there is a $G \in \mathscr{D}_t$ such that $H \cup G$ contains an irreducible sub-arc of A from t' to t and $(H \cap G)\backslash A$ is a topological ray r such that $\bar{r}\backslash r=u$ is a point of A, $r \cap \partial G = r \cap \partial H$ is the unique end-point of r.

Suppose the last point of A in G after t is z, $t<z$. Then let F be the component of $H\backslash \bar{r}$ determined by $[0, u]$ and F_1 the component of $G\backslash \bar{r}$ determined by $(u, z]$. Then the closure of $F \cup F_1$ is a 2-cell whose boundary contains $[0, z]$. Since t is interior to $[0, z]$, the set X is open.

Let $t_1 < t_2 < \cdots$, where $t_n \in X$ and $\lim t_n = t$. Since A is LU at t, there is a disk D_t whose boundary contains a neighborhood of t in A. Thinking of D_t as an element of $\mathscr{D}_t$ and using the strong local unknottedness property, some t_n determines a $\mathscr{D}_{t_n}$ that is contiguous to $\mathscr{D}_t$. Thus there is a pair of disks D_t, D_{t_n} that can be joined as above to give a disk E whose boundary contains the arc

$[0, t]$. Since X is open and closed in the connected set A, there is a disk E^* whose boundary contains A.

Proof of Theorem β. If J is the simple closed curve in question, we proceed as above to find a surface S obtained by pasting a pair of disks together along opposite edges. If S is either an annulus or Moebius strip, the desired R is easily found.

To find the surface S we start by writing $J=J_1\cup J_2$ where J_1 and J_2 are non-overlapping arcs with end-points a and b. Since J_1 is SLU and is LU at a and b, Theorem α applies and these is a disk E whose boundary contains J_1 that is disjoint to the interior of J_2. Let D_a be a disk whose boundary contains a neighborhood of a. We suppose D_a is sufficiently small that for arbitrary $u, v \in J\cap\partial D_a$, $\mathscr{D}_u$ and $\mathscr{D}_v$ are contiguous. Suppose b was chosen sufficiently near a that $\mathscr{D}_a$ and $\mathscr{D}_b$ are contiguous. Since the arc ab of J is now not unique we are in case β of the definition of SLU. Thinking of E as an element of $\mathscr{D}_{x'}$ for $x'<a$ for x' arbitrarily near a, and using the fact that $\mathscr{D}_{x'}$ and $\mathscr{D}_b$ are contiguous, there is a disk $G\in\mathscr{D}_b$, such that

$$(E\cap G)\backslash J$$

is a pair of disjoint topological rays r and r_1, each with one limit point on J. Note $q\neq q_1$.

Then $\bar{r}$ and $\bar{r}_1$ separate E into three components and one, call it E_1 contains $J\backslash(J\cap\partial G)$ on its boundary. Also, $\bar{r}$ and $\bar{r}_1$ separate G into three components, one of which, call it G_1 contains a on its boundary. Then E_1 and G_1 are disks. The closure of the union of E_1 and G_1 obtained by identification along $\bar{r}$ and $\bar{r}_1$ is the desired surface S.

Strongly locally unknotted curves lie on solid tori.

Let k be a boundary component of an annulus A that is locally polyhedral modulo k. Topologically $A=S^1\times I$ is embedded in three-space and we wish to show that A can be fattened to a solid torus with k on its boundary. That is, we want to construct a set $S^1\times I^2=M$ in three-space with k on its boundary. *Rosen's* thesis [18] shows that a topological torus that is locally tame in E^3 except at exactly one point *may* have neither complementary domain with a closure a solid torus. Thus, if a torus is (possibly) wild at a simple closed curve of points, it is not clear such a torus has a complementary domain whose closure is a solid torus.

First of all, consider the construction of M. Although the main idea here is very simple, the precise equations are very laborious and have been written out in detail for a similar procedure in [13, pages 16–17], hence we will only sketch the main points.

Let $f\colon S^1\times I\to A$ where $f|S^1\times(0)\to k$. We imagine A is parameterized by the homeomorphism f and write $I=\{0\}\cup\bigcup_{j=1}^{\infty} I_j$, where $I_j=[1/(j+1), 1/j]$. Each $f[S^1\times I_j]$ is a polyhedral annulus and can be swelled to form a polyhedral solid torus M_j that contains $f[S^1\times I_j]$ as a longitudinal annulus.

We suppose $M_j\cap k=\phi$, $M_{j-1}\cap M_j$ is an annulus, $M_j\cap M_k=\phi$ if $|j-k|>1$, see Figure 1.

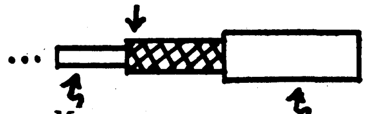


FIG. 1.

Let $F(x, u, t)$ be the extension of $f(x, u)$ from $S^1\times I$ to $S^1\times I^2$ such that

$$F(x, u, 0)=f(x, u)$$

and t varies from -1 to $+1$ in the second factor I.

Let $\varepsilon_1>\varepsilon_2>\varepsilon_3\cdots$ be positive numbers such that $\sum \varepsilon_K<\infty$. For each $j=1, 2, \cdots$ we cut M_j into cubes of diameter $<\varepsilon_j$ by meridional, disjoint disks. Denote these cubes by $C_j', \cdots, C_j^{n_j}$. By inserting these disks properly we can arrange that

$$C_j^{\lambda}\cap C_{j+1}^{\mu}=\phi\ , \qquad \text{or a disk}\ ,$$

that we need not give a name to at present.

From the construction, if $l_1, l_2, \cdots$ is a sequence of integers such that

$$C_i^{l_i}\cap C_{i+1}^{l_{i+1}}\neq\phi\ ,$$

then $\bigcup_1^N C_i^{l_i}$ is a topological cube diameter $<\sum_1^N \varepsilon_i$. This may be seen as follows.

By the condition $\sum\varepsilon_i<+\infty$, the closure of $\bigcup_{j=1}^{\infty} C_j^{l_j}$ has at most one limit point on k, hence the closure is a closed 3-cell with precisely one point on k. To verify this, suppose $\bigcup_{j=1}^{\infty} C_j^{l_j}$ has at least two limit points p and q on k. Let $V(p)$ and $V(q)$ be neighborhoods of p and q respectively with disjoint closures. There is a positive number $y>0$ such that any connected set meeting both $V(p)$

and $V(q)$ has a diameter $\geq y$. By the choice of $l_1, l_2, \cdots$ there is a sequence of integers $N_1, N_1', N_2, N_2', \cdots$ such that the last element of $C_1, \cdots, C_{N_1}$ is in $V(p)$ and the last element of $C_1, \cdots, C_{N_1'}$ is in $V(q)$ and

$$N_1 < N_1' < N_2 < N_2' < N_3 < N_3' \cdots .$$

Since the diameter of the union of two connected sets having a point in common is not more than the sum of the diameters of the two sets, we have

$$y \leq \operatorname{diam} (C_{N_1+1} \cup \cdots \cup C_{N_1'}) \leq \sum_{N_1+1}^{N_1'} \operatorname{diam} C_{N_j}$$

and

$$y \leq \operatorname{diam} (C_{N_1'+1} \cup \cdots \cup C_{N_2}) \leq \sum_{N_1'+1}^{N_2} \operatorname{diam} C_{N_j}$$

or

$$2y \leq \sum_{N_1+1}^{N_2} \varepsilon_j .$$

Hence

$$2\lambda y \leq \sum_{N_1+1}^{N_1+\lambda} \varepsilon_j ,$$

which denies that $\sum \varepsilon_j < +\infty$. Thus $\bigcup_1^\infty C_j^{l_j}$ has exactly *one* limit point on k.

From this closed 3-cell, C, one can easily find a disk D whose boundary encircles $f[S^1 \times I]$ and has exactly one point on k.

Let M=closure $\bigcup_{j=1}^\infty M_j$. The existence of a pair of disjoint D's as above permits us to split M into a pair of closed 3-cells meeting only along the disks (disjoint) D_1 and D_2, hence M is a solid torus (not generally tame).

Suppose *Fox*'s "Remarkable Simple Closed Curve" is SLU. Then by Theorem α, above, denoting this curve by k, there is an annulus A with boundary components k and l. Let $l_1, l_2, \cdots$, be a sequence of polygonal curves on $A \backslash k$ such that $\lim l_n = k$, where the l's are disjoint and convergence is in the sense $\max_{x \in S_1} \rho[l_n(x), k(x)] = \operatorname{dist}(l_n, k)$, ρ denoting the metric of E^3.

Now the crookedness in the sense of *Milnor* of k, $\mu(k) = +\infty$. Since each l_n is tame, $\mu(l_n) < +\infty$. Since l_n and l_m are of the same isotopy class, $\mu(l_n) = \mu(l_m)$, n, m arbitrary, where $\mu(l_n)$=crookedness of isotopy class of l_n [17]. But the number of maxima of the function $\boldsymbol{b} \cdot \boldsymbol{r}(t)$ for the unit vector $\boldsymbol{b}$ clearly increases indefinitely as $n \to \infty$. Hence the hypotheses $\mu(l_n) = \mu(l_m)$ leads to a contradiction We conclude k lies on no such annulus. That is, k is not SLU.

The core of the solid torus is defined by

$$F(S^1, \tfrac{1}{2}, 0) = m .$$

where F is as on page 56. The embedding of a polyhedral annulus N is determined by the embedding of its 'core' and by its twist $\rho=\varepsilon\cdot v(m, l)$, where m is the median and v=linking number of m, l $l=l_1+l_2$, $l_1 \sim l_2$, l_1 and l_2 being the components of the boundary of N [18]. In our case we can think of l_2 as being arbitrarily close to k. In any case there is a well-defined integer $g(m)$ the genus of the tame knot m and the least value of $g(m)$ for different choices of A is an invariant of the embedding of k. Let $\alpha(k)$ be the minimum. If k is tame, $\alpha(k)$ is the classical genus of k. If $\mathscr{K}$ is a class of wild simple closed curves for which a classical genus is assigned, does $\alpha(k)$ equal the classical genus?

Some unsolved problems (summary).

Give an example of a 1-manifold that is LU but not SLU.

If a curve k is locally unknotted at each point, is it the boundary of an orientable surface?

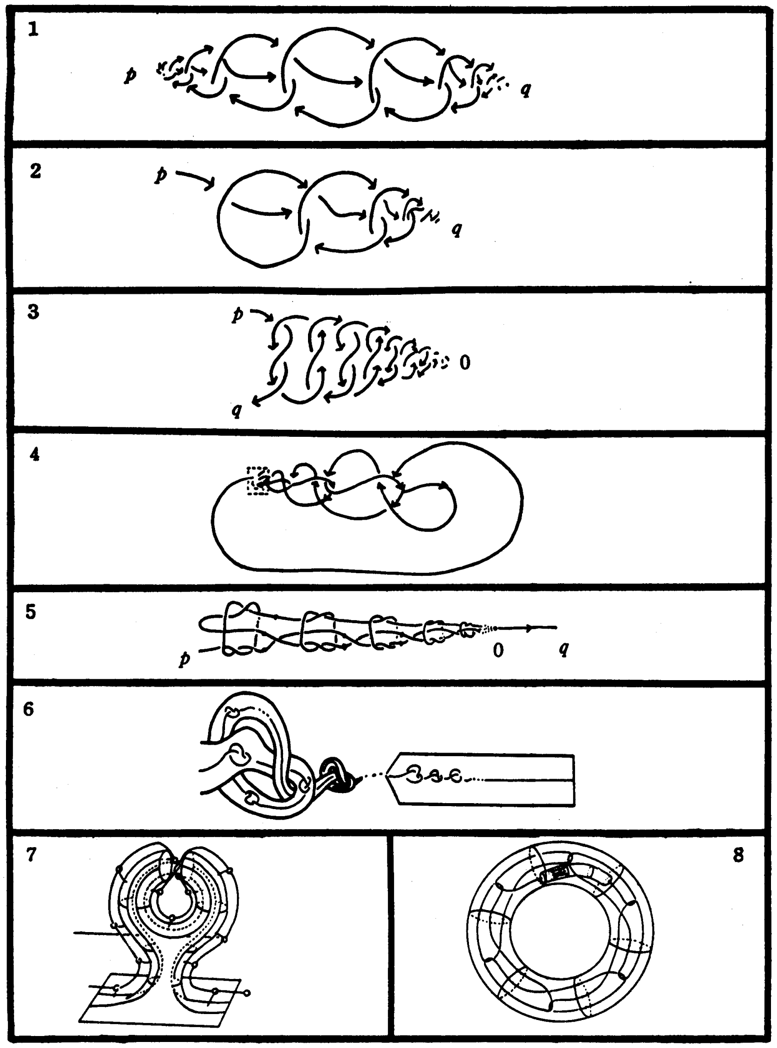
If the curve k is locally unknotted and is locally tame except for one point, what additional hypotheses are necessary to determine the equivalence type of k?

A complete set of invariants of the wild simple closed curves are known if k is locally tame except at one point and if k is LPU at all points (see *Fox-Harrold* [9]).

The next simplest case would seem to be a curve k that is LT mod p (p a point) and k is LU. Thus k could fail to be LPU only at p.

Conjecture

Suppose k and l bound an annulus A which is locally polyhedral mod k that is *untwisted* and *unknotted, i.e.,* l_n and l bound a polyhedral annulus (see page 23) that is untwisted and unknotted. Then k is the boundary of an orientable surface.



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