ON THE MAXIMUM MODULUS AND THE MEAN VALUE OF AN ENTIRE FUNCTION OF TWO COMPLEX VARIABLES

By

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1. Let

$$f(z_1, z_2)=\sum_{k_1,k_2=0}^{\infty} a_{k_1k_2} z_1^{k_1} z_2^{k_2} \tag{1.1}$$

be a transcendental entire function of the two complex variables z_1, z_2. Following the notation of [1] and [2], let

$$M_\delta(r_1, r_2)=\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi} |f(r_1e^{i\theta_1}, r_2e^{i\theta_2})|^\delta d\theta_1 d\theta_2\right\}^{1/\delta}, \qquad \delta\geq 1 ,$$

$$M(r_1, r_2)=\max_{|z_j|\leq r_j} |f(z_1, z_2)| , \qquad j=1, 2 ,$$

$$m(r_1, r_2)=\sum_{k_1,k_2=0}^{\infty} |a_{k_1k_2}| r_1^{k_1} r_2^{k_2} ,$$

$\mu(r_1, r_2)$ and $\nu(r_1, r_2)$ denotes the maximum term and the rank of the maximum term of the series expansion of the function $f(z_1, z_2)$ given by (1.1). The finite order ρ of the function $f(z_1, z_2)$ is defined as [[3], p. 218]

$$\limsup_{r_1,r_2\to\infty} \frac{\log \nu(r_1, r_2)}{\log (r_1 r_2)}=\rho . \tag{1.2}$$

In this paper we have extended the results due to *Brinkmeier* [[4], Satz 28] and the well-known result due to *Valiron* [5] to the case of functions of two complex variables.

Lemma 1. *If the function* $f(z_1, z_2)$ *has a series representation* (1.1), *then*

$$\{M_2(r_1, r_2)\}^2=\sum_{k_1,k_2=0}^{\infty} |a_{k_1k_2}| r_1^{2k_1} r_2^{2k_2} .$$

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Proof.

$$|f(z_1, z_2)|^2 = \sum_{k_1,k_2=0}^{\infty} a_{k_1k_2} z_1^{k_1} z_2^{k_2} \sum_{m_1,m_2=0}^{\infty} \overline{a_{m_1m_2}} \bar{z}_1^{m_1} \bar{z}_2^{m_2}$$

$$= \sum_{k_1,k_2=0}^{\infty} |a_{k_1k_2}|^2 r_1^{2k_1} r_2^{2k_2} + \sum_{\substack{k_j \neq m_j \\ (j=1,2)}} a_{k_1k_2} \overline{a_{m_1m_2}} r_1^{k_1} \bar{r}_1^{m_1} r_2^{k_2} \bar{r}_2^{m_2} e^{i\{\theta_1(k_1-m_1)+\theta_2(k_2-m_2)\}} .$$

On integrating term by term, the result follows.

Theorem 1. *If $f(z_1, z_2)$ is an entire function of finite order ρ, then*

$$m(r_1, r_2) \leq M_2(r_1, r_2)(r_1 r_2)^{(1/2)\rho+\varepsilon} ,$$

for all $r_1 > r_1^\circ(\varepsilon)$ and $r_2 > r_2^\circ(\varepsilon)$, ε being any arbitrary small number.

Proof. Let $0 < r_j < R_j$ and $p_j \leq k_j \geq 0$ $(j=1, 2)$. We know

$$m(r_1, r_2) = \left\{ \sum_{k_1,k_2=0}^{p_1-1,p_2-1} + \sum_{k_1=p_1,k_2=p_2}^{\infty} \right\} |a_{k_1k_2}| r_1^{k_1} r_2^{k_2}$$

$$< \{\nu(r_1, r_2)\}^{1/2} \left\{ \sum_{k_1,k_2=0}^{p_1-1,p_2-1} |a_{k_1k_2}|^2 r_1^{2k_1} r_2^{2k_2} \right\}^{1/2}$$

$$+ \mu(r_1, r_2) \sum_{k_1=p_1,k_2=p_2}^{\infty} \frac{|a_{k_1k_2}| R_1^{k_1} R_2^{k_2}}{|a_{p_1p_2}| R_1^{p_1} R_2^{p_2}} \left\{ \frac{r_1}{R_1} \right\}^{k_1-p_1} \left\{ \frac{r_2}{R_2} \right\}^{k_2-p_2}$$

Using Holder's Inequality [[2], pp. 66–61]

$$\leq \{\nu(r_1, r_2)\}^{1/2} M_2(r_1, r_2) + \mu(r_1, r_2) \frac{R_1 R_2}{(R_1 - r_1)(R_2 - r_2)}$$

Using Lemma 1

$$\leq M_2(r_1, r_2) \left[\{\nu(r_1, r_2)\}^{1/2} + \frac{R_1 R_2}{(R_1 - r_1)(R_2 - r_2)} \right].$$

Using (1.2) and letting $R_j = r_j$ $(j=1, 2)$, we have

$$m(r_1, r_2) \leq M_2(r_1, r_2) \ (r_1 r_2)^{(1/2)\rho+\varepsilon} .$$

Theorem 2. *If $f(z_1, z_2)$ is an entire function of finite order ρ, then*

$$M_2(r_1, r_2) \leq \mu(r_1, r_2) \ (r_1 r_2)^{(1/2)\rho+\varepsilon} ,$$

for all $r_1 > r_1^\circ(\varepsilon)$ and $r_2 > r_2^\circ(\varepsilon)$.

Proof. Let $0 < r_j < R_j$ and $p_j \leq k_j \geq 0$ $(j=1, 2)$. From Lemma 1, we have

$$\{M_2(r_1, r_2)\}^2 = \left\{ \sum_{k_1,k_2=0}^{p_1-1,p_2-1} + \sum_{k_1=p_1,k_2=p_2}^{\infty} \right\} |a_{k_1k_2}|^2 r_1^{2k_1} r_2^{2k_2}$$

$$\leq \nu(r_1, r_2) \{\mu(r_1, r_2)\}^2 \left[1 + \sum_{k_1=p_1,k_2=p_2}^{\infty} \frac{|a_{k_1k_2}| R_1^{2(k_1-p_1)} R_2^{2(k_2-p_2)}}{|a_{p_1p_2}|} \right.$$

$$\cdot\left\{\frac{r_1}{R_1}\right\}^{2(k_1-p_1)}\left\{\frac{r_2}{R_2}\right\}^{2(k_2-p_2)}\Bigg]$$

$$\leq\nu(r_1, r_2)\{\mu(r_1, r_2)\}^2\left[1+\frac{R_1^2R_2^2}{(R_1^2-r_1^2)(R_2^2-r_2^2)}\right].$$

Using (1.2) and letting $R_j=r_j$ $(j=1,2)$, we have

$$M_2(r_1, r_2)\leq\mu(r_1, r_2)(r_1r_2)^{(1/2)\rho+\varepsilon}$$

Theorem 3. *If $f(z_1, z_2)$ is an entire function of finite order ρ, then*

$$m(r_1, r_2)\leq\mu(r_1, r_2)(r_1r_2)^{\rho+\varepsilon}.$$

The theorem follows by combining Theorems 1 and 2.

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