CONTRACTIBLE NEIGHBORHOODS IN MANIFOLD FACTORS

By

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1. Introduction.

In the paper "Concerning suspension spheres" [8], the author showed in Theorem 4 that if Z is a compact metric space of the homotopy type of S^n so that $Z\times R^1$ is an open $(n+1)$-manifold, $n\geq 4$ and Z contains a topological n-cell, then the suspension of Z is a topological $(n+1)$-sphere. In this paper we investigate some related suspension problems. Along the way we shall prove, by somewhat simpler means, a more general result than the above; namely, Theorem 1: Let (X, B) be a collared compact metric pair. If $(X-B)\times R^k$ is an open $(n+k)$-manifold, $n+k\geq 5$ and $k>0$, and X is contractible, then the suspension of $X\times I^k$ is homeomorphic to I^{n+k+1}.

The notation X^*Y represents the join of spaces X and Y. The suspension of X is $S^{0*}X=S(X)$; the cone of X is $I^{0*}X=C(X)$. $OC(X)=C(X)-X$ is called the open cone over X. A closed pair (X, A) is called *collared* (*bicollared*) if A has an open neighborhood N for which $(N, A)\approx(A\times[0, 1), A\times 0)$ $((N, A)\approx(A\times(-1, 1), A\times 0))$.

Suppose (X, A) and (Y, B) are disjoint closed pairs and $f: A\approx B$ is a bicontinuous bijection. $(X, A)\#(Y, B)$ denotes the adjunction space $X\bigcup_f Y$. If (Y, B) is a disjoint copy of (X, A) then $2(X, A)=(X, A)\#(Y, B)$ is called the *double of* (X, A). In some cases where little confusion will result we shall simply write $X\# Y$ or $2X$, as for example when doubling a manifold with boundary along its boundary.

2. Suspending contractible neighborhoods.

Henceforth all spaces considered are T_2.

Lemma 1. *Let (X, B) be a finite dimensional compact collared pair. Suppose X is acyclic with respect to Čech cohomology and $X-B$ is an n-gm, a generalized n-manifold in the sense of Čech cohomology. Then B is an $(n-1)$-gm with the*

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cohomology of S^{n-1}.

Proof. $B\times R^1$ is homeomorphic to an open subset of $X-B$, so that $B\times R^1$ is an orientable n-*gm*. By Theorem 6 of [5] B is an $(n-1)$-*gm* and by the Kunneth formula B is orientable. Since B is collared in X, the latter is an n-*gm* with boundary B as defined in [5; 1.3]. Thus X is a generalized n-cell in the sense of [2; 1.1] so that B has the cohomology of S^{n-1}.

We thank *F. Reymond* for helping to simplify this proof.

Theorem 1. *Let* (X, B) *be a collared compact metric pair. Suppose that* $(X-B)\times R^k$ *is an open* $(n+k)$-*manifold,* $n+k\geq 5$ *with* $k>0$, *and* X *is contractible. If* $Y=X\times I^k$, *then* $Y\approx C(\partial Y)$ *and* $S(Y)\approx I^{n+k+1}\approx Y\times I$.

Proof. Consider $Y=X\times I^k$. Lemma 2 of [8] indicates that $(X\times I, \partial(X\times I))$ is a collared pair, so that inductively it follows that $(Y, \partial Y)$ is a collared pair.

If $n=1$, $X-B\approx R^1$ and Int $Y\approx R^{k+1}$. If $n>1$ thentby lemma 1 B is connected. This implies that $X-B$ is 0-connected at infinity so that $(X-B)\times R^k$ is 1-connected at infinity. Applying Theorem 1.1 of [9] we again may conclude that $(X-B)\times R^k=\text{Int } Y\approx R^{n+k}$.

Since $(Y, \partial Y)$ is collared, the compact space $2(Y, \partial Y)$ is covered by two open $(n+k)$-cells. Therefore by Theorem 3 of [6], $2Y\approx S^{n+k}$. Now Lemma 3 of [8] suggests that $S(\partial Y)\approx S^{n+k}$ and Int $Y\approx OC(\partial Y)$. This means that $(Y, \partial Y)\approx(C(\partial Y), \partial Y)$.

This leads immediately to the formulas $S(Y)\approx S(C(\partial Y))\approx C(S(\partial Y))\approx C(S^{n+k})\approx I^{n+k+1}$.

Finally $Y\times I\approx C(\partial Y)\times C(S^0)$. Lemmas 3 and 4 of [4] then give us the chain of homeomorphisms, $C(\partial Y)\times C(S^0)\approx C(S(\partial Y))\approx C(S^{n+k})\approx I^{n+k+1}$.

We next examine an easy application of our work to polyhedral neighborhoods in triangulated manifolds.

Theorem 2. *Let* K *be a compact contractible subcomplex interior to the metric triangulated* n-*manifold* M. *Then if* X *is a closed 2nd-derived neighborhood of* K *in* M *and* $n\geq 5$, $2X\approx S^n$.

Proof. Let $B=\text{Bd}\, X$ in M. From Lemmas 1 and 3 of [7] it follows that (X, B) is a collared pair, $X\sim K$ and $2(X, B)$ is an n-manifold. The pair (X, B) satisfy the hypotheses of Theorem 1, so that $S(2X)\approx S(\partial(X\times I))\approx S^{n+1}$. Consequently $2X$ is a closed n-manifold of the homotopy type of S^n and is therefore a topological n-sphere by Theorem 7 of [3].

Corollary. *With the same hypotheses,* $X\times I\approx I^{n+1}$.

Proof. This is a direct result of Theorem 1 of [8].

Note. For $n \geq 5$, this generalizes Theorems 1 and 2 of [7].

Theorem 3. *(Desuspension Theorem) Let (X, B) be a collared compact pair so that $S(B) \approx S^{n-1}$ and $S(X \times I) \approx I^{n+1}$. Then $S(X) \approx I^n$.*

Proof. It follows that $S^n \approx S(\partial(X \times I)) \approx S(2X)$. Inasmuch as B is bicollared in $2(X, B)$, if we refer to Lemma 4 of [8] we may see that $S(X) \approx I^n$.

Application. We shall show, as was claimed in the Introduction, that Theorems 1 and 3 above imply Theorem 3 of [8], which in turn is used to prove Theorem 4 of [8].

Thus let (X, A) be a collared compact metric pair so that $S(A) \approx S^n$, and $(X-A) \times R^1$ is an open $(n+1)$-manifold. Furthermore X is contractible and $n \geq 4$. By Theorem 1 $S(X \times I) \approx I^{n+2}$. Now if we use the Desuspension Theorem, we may deduce that $S(X) \approx I^{n+1}$.

3. Some other suspension results.

Definition. If $x \in X$ the reduced suspension $RS(X, x)$ will be the quotient space $(S^{0*}X)/(S^{0*}x)$.

In the terminology used by the author in [8] our next result may be described as follows. If X is a suspension sphere and $v \in X$ has a cone neighborhood C, then $2(X - \mathrm{Int}\, C)$ is a double suspension sphere.

Theorem 4. *Let $S(X) \approx S^n$ and suppose $v \in X$ has a closed neighborhood C with the following properties. $(C, \mathrm{Bd}\, C, v) \approx ((\mathrm{Bd}\, C)^* v, \mathrm{Bd}\, C, v)$ and $\mathrm{Bd}\, C$ is bicollared in X. Then if $Y = 2(X - \mathrm{Int}\, C)$, $SS(Y) = S^{1*}Y \approx S^{n+1}$. Also if $n \neq 4$ and $\pi_1(Y) = 0$, $S(Y) \approx S^n$ and $RS(X, v) \approx S^n$.*

Proof. We represent S^0 as $\{p, q\}$ so that $S^n \approx S(X) = p^*X^*q$ and denote p^*v^*q by K. If we properly fit together the linear parameters in $S(X)$ and C, we may find a closed neighborhood M of K in $S(X)$ for which $(M-K, \mathrm{Bd}\, M) \approx (Y \times [0, 1), Y \times 0)$ and $\mathrm{Bd}\, M$ is bicollared in $S(X)$.

More precisely let the cones p^*X, X^*q and $(\mathrm{Bd}\, C)^*v = C$ be represented as quotient spaces of $X \times [-1, 0]$, $X \times [0, 1]$ and $(\mathrm{Bd}\, C) \times [0, 1]$, respectively, with quotient maps g_i, $i = 1, 2, 3$, so that $g_1(X \times -1) = p$, $g_2(X \times 1) = q$ and $g_3((\mathrm{Bd}\, C) \times 1) = v$. Then we may choose M to be the set $g_1(X \times [-1, -\frac{1}{2}]) \cup p^*g_3((\mathrm{Bd}\, C) \times [\frac{1}{2}, 1])^*q \cup g_2(X \times [\frac{1}{2}, 1])$ and it is fairly clear that $\mathrm{Bd}\, M \approx \partial[(X - g_3((\mathrm{Bd}\, C) \times (\frac{1}{2}, 1])) \times I] \approx Y$.

Now $Z = S^n/K$ suspends to S^{n+1} [1]. Let $f: S^n \to Z$ be the natural quotient map so that $f(K) = k$. It may be seen that $f(M) \approx Y^*k$ is a neighborhood of k in

Z. Accordingly $S(f(M))$ is a neighborhood of k in $S(Z)\approx S^{n+1}$. Since $S(f(M)), k)\approx (S(Y^*k), k)\approx(S(Y)^*k, k)$, then by Theorem 4 of [6] $S(S(Y))\approx S^{n+1}$.

Now assume $\pi_1(Y)=0$ and $n\neq 4$. If $n<4$, X and Y are spheres. If $n\geq 5$ $\operatorname{Int} M$ is contractible and 1-connected at infinity hence $\operatorname{Int} M\approx R^n$ [9]. From Lemma 3 of [8] it now follows that $S(\operatorname{Bd} M)\approx S(Y)\approx S^n$. Thus $S^n\approx S(X)/K=RS(X, v)$.

Corollary. *Let X be a polyhedron which suspends to an n-sphere and v be a vertex of X. If L is a 1st-derived link of v in X, $SS(2(X-\operatorname{Int}(L^*v)))\approx S^{n+1}$.*

Proof. For then L is bicollared in X and we may simply apply the theorem.

Turning for a moment to the category of PL manifolds we derive a very simple characterization of cellular subpolyhedra in manifolds of sufficiently high dimension.

Theorem 5. *Let K be a subpolyhedron interior to the PL n-manifold M with $n\geq 6$. If N is a closed 2nd-derived neighborhood of K in M, K is cellular in M if and only if N is a PL n-cell.*

Proof. N is a closed mapping cylinder from ∂N over K. Consequently if $N\cong I^n$, K is cellular by Theorem 6 of [6].

Assume that K is cellular in M. The Theorem quoted in the first case tells us that $\operatorname{Int} N\approx R^n$ and the PL manifold $\partial N\sim S^{n-1}$. An application of Smale's Generalized Poincare Theorem [10] allows us to assert that $N\cong I^n$.

4. An elementary suspension theorem and some questions.

Our next result is so basic that it has undoubtedly been noticed by others; nevertheless it can be used in establishing certain fundamental properties of manifolds. Rather than give any applications, of which some will be clear to the reader, we are more interested in raising some questions about possible generalizations.

Theorem 6. *Let X and Y be compact connected spaces for which $X\times R^1\approx Y\times R^1$. Then $S(X)\approx S(Y)$.*

Proof. Let $h: X\times R^1\approx Y\times R^1=Z$ and $X_t=X\times t$, $Y_t=Y\times t$. By the compactness of X there exist real numbers $t_1<t_2$ so $h(X_0)\subseteq Y\times(t_1, t_2)$. Let $U_1=h(X\times(-\infty, 0))$, $U_2=h(X\times(0, \infty))$, $K_1=Y\times(-\infty, t_1]$ and $K_2=Y\times[t_2, \infty)$. Obviously $K_1\cup K_2\subseteq Z-h(X_0)$.

Each connected set K_i lies in one open set U_j, $i, j=1, 2$. Assume that both K_1 and K_2 lie in U_i where (i, j) is a permutation of $(1, 2)$. Then $\operatorname{Int} K_1\cup\operatorname{Int} K_2\approx U_i$

which means that $\mathrm{Cl}\, U_j(\approx X\times[0, \infty))\subseteq Y\times[t_1, t_2]=Z-(\mathrm{Int}\, K_1\cup \mathrm{Int}\, K_2)$. This indicates that $\mathrm{Cl}\, U_j$ is compact, which is absurd. Hence $K_1\subseteq U_i$ and $K_2\subseteq U_j$. Necessarily then the one point compactification of K_1 compactifies $\mathrm{Cl}\, U_i$; similarly for K_2 and $\mathrm{Cl}\, U_j$. Now the conclusion easily follows.

Questions. Theorem 6, like Theorem 1, is somewhat unsatisfactory in that it only deals with single suspensions, that is joins with S^0. An interesting generalization would be found if one could place nonobvious conditions on X and Y so that $X\times R^k\approx Y\times R^k$ would imply that $X^*S^{k-1}\approx Y^*S^{k-1}$. An obvious condition of this sort would be to demand that $OC(X)\times R^{k-1}\approx OC(Y)\times R^{k-1}$. This may be seen by again applying Noguchi's Lemmas 3 and 4 of [4] and the facts that $R^{k-1}=OC(S^{k-2})$ and if Z is compact then $C(Z)/Z\approx S(Z)$.

From the Application in section 2 it may be seen that for certain pairs (X, B) Theorems 1 and 3 are sufficient to imply that $X^*S^k\approx I^{n+k+1}$. Still this method is not adequate to deal with a pair (X, B) as in Theorem 1 with $B\sim S^{n-1}$. One might still hope to develope combinatorial methods which would cope with this case.

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