THE STONE-ČECH COMPACTIFICATION OF A BASICALLY DISCONNECTED SPACE

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1. Introduction. We prove the following result:

Theorem. *Let S be a locally compact Hausdorff space that is basically disconnected but not countably compact. Then*

(a) *$\beta S-S$ contains $\mathbf{c}$ pairwise disjoint nonempty clopen sets, and*
(b) *$\beta S-S$ is not basically disconnected.*

This generalizes the well-known result that (a) and (b) hold when S is the space of integers (cf. [3, 6S, 6W]).

2. Background. Let S be a completely regular space.

A subset of S that is both open and closed is *clopen.*

A set of the form $\{p \in S : f(p) \neq 0\}$ for some continuous scalar-valued function f on S is a *cozero set* of S. Note that a countable union of cozero (clopen) sets is a cozero set [3, 1.14].

S is *basically disconnected* if the closure of every cozero set is open. See [3] and [5] for properties of basically disconnected spaces. It is proven in the latter reference that they are characterized by the following condition:

$$\mathrm{cl}\,(U \cap V) = \mathrm{cl}\, U \cap \mathrm{cl}\, V\,.$$

for every cozero set U and every open set V.

$\mathbf{c}$ denotes the cardinality of the continuum, and N the positive integers.

3. Proof of the Theorem. Let S satisfy the hypothesis. Since S is not countably compact it has some closed denumerable discrete subspace $\{p_n\}_{n \in N}$. Since S is locally compact and has a base of clopen sets, we can choose a sequence $\{V_n\}_{n \in N}$ of pairwise disjoint compact open subsets of S such that $p_n \in V_n$ for all $n \in N$.

For each subset A of N, we define

$$A' = (\beta S - S) \cap \mathrm{cl}_{\beta S} \bigcup_{n \in A} V_n\,.$$

We claim that the following hold for every $A, B \subset N$.

(1) A' is clopen in $\beta S - S$.

(2) $(A \cup B)' = A' \cup B'$.

(3) $A' \neq \phi$ if and only if A is infinite.

(4) If $A \subset B$ then $A' \subset B'$.

(5) If $A \cap B = \phi$ then $A' \cap B' = \phi$.

(6) $A' \subset B'$ if and only if $A - B$ is finite.

(7) $A' = B'$ if and only if $(A-B) \cup (B-A)$ is finite.

(8) $A' \cap B' = \phi$ if and only if $A \cap B$ is finite

(1) follows from the fact that $\bigcup_{n \in A} V_n$ is a cozero set of the basically disconnected space βS and (2) is trivial.

To prove (3), suppose A is infinite. Then $\{p_n\}_{n \in A}$ is infinite, and since it is also discrete and closed in S,

$$(\beta S - S) \cap \mathrm{cl}_{\beta S} \{p_n\}_{n \in A}$$

is a nonempty subset of A'.

Conversely if A is finite, then

$$\mathrm{cl}_{\beta S} \bigcup_{n \in A} V_n = \bigcup_{n \in A} V_n \subset S,$$

so $A' = \phi$.

(4) is trivial and (5) holds since disjoint cozero sets of βS have disjoint closures.

To prove (6), suppose $A - B$ is finite. Then $(A-B)' = \phi$ by (3). Since $(A \cup B)' = B' \cup (A-B)'$ by (2), we have $A' \subset (A \cup B)' = B'$.

Conversely, if $A - B$ is infinite, then $\phi \neq (A-B)' \subset A'$. Since $(A-B)' \cap B' = \phi$ by (5), it follows that A' is not a subset of B'.

(7) follows directly from (6).

To prove (8), suppose $F = A \cap B$ is finite. Then $A' = (A-F)'$ and $B' = (B-F)'$ by (7). But $(A-F)' \cap (B-F)' = \phi$ by (5). Hence A' and B' are disjoint.

Conversely, if $A \cap B$ is infinite, then $(A \cap B)'$ is a nonempty subset of $A' \cap B'$ by (3).

Now, if $\mathfrak{F}$ is any family of $\boldsymbol{c}$ infinite subsets of N, the intersection of any two of which is finite (see, e.g., [3, 6Q]), then $\{A' : A \in \mathfrak{F}\}$ satisfies the requirements of (a).

To prove (b), choose a strictly increasing sequence $\{A'_n\}$ and let C be an arbitrary clopen subset of $\beta S - S$ containing $\bigcup_{n=1}^{\infty} A'_n$. Since βS has a base of clopen sets, there exists a clopen subset G of βS such that $C=(\beta S-S)\cap G$ (c.f. [4, p. 75, Lemma 2]). Let $E=G\cap S$. Then E is clopen in S and since S is open and dense in βS, is follows that

$$C=(\beta S-S)\cap \mathrm{cl}_{\beta S} E .$$

Let $M=\{k\in N: p_k\in E\}$, and for each subset B of M, define

$$B''=(\beta S-S)\cap \mathrm{cl}_{\beta S}\bigcup_{k\in B}(V_k\cap E) .$$

We claim the following hold for every $B\subset M$.

(i) $B''=B'\cap C$.

(ii) B'' is clopen in C.

(iii) If $B'\subset C$ then $B'=B''$.

To prove (i), note that

$$\mathrm{cl}_{\beta S}\bigcup_{k\in B}(V_k\cap E)=\mathrm{cl}_{\beta S} E\cap \mathrm{cl}_{\beta S}\bigcup_{k\in B} V_k ,$$

since E is open and $\bigcup_{k\in B} V_k$ is a cozero set of the basically disconnected space βS.

(ii) and (iii) follow immediately from (i).

We also need the following.

(iv) If $A\subset N$ and $A'\subset C$ then $A'=(A\cap M)'=(A\cap M)''$.

To prove this, suppose $D=A-(A\cap M)$ were infinite. Then $p_k\notin E$ for all $k\in D$. Therefore,

$$\phi=\mathrm{cl}_{\beta S} E\cap \mathrm{cl}_{\beta S}(S-E)\supset \mathrm{cl}_{\beta S} E\cap \mathrm{cl}_{\beta S}\{p_k\}_{k\in D} ,$$

so C and $\mathrm{cl}_{\beta S}\{p_k\}_{k\in D}$ are disjoint. But $\{p_k\}_{k\in D}$ is infinite, discrete, and closed in S. Since βS is compact,

$$(\beta S-S)\cap \mathrm{cl}_{\beta S}\{p_k\}_{k\in D}$$

is a nonempty subset of A', and hence of C. This contradiction proves that $A'=(A\cap M)'$. The remainder of (iv) now follows from (iii).

It is now easily verified that for every $A, B\subset M$, the properties (1) through (8) hold if A' and B' are everywhere replaced by A'' and B''.

We next let $B_n=A_n\cap M$ for every n. It follows from (iv) that $B_n''=A_n'$. Therefore, $\{B_n''\}$ is strictly increasing, M'' is a clopen subset of $\beta S-S$, and

$$\bigcup_{n=1}^{\infty} B_n''\subset M''\subset C .$$

It follows from (4) and (2) that for each n there exists $t_n \in B_n - \bigcup_{k=1}^{n-1} B_k$. Letting $F = M - \{t_n\}_{n\in N}$, we have $F'' \subset M'' \subset C$. Hence $\bigcup_{n=1}^{\infty} B_n'' \subset F''$ and $F'' \neq M''$. Now, $\mathrm{cl}_{\beta S-S} \bigcup_{n=1}^{\infty} A_n' = \mathrm{cl}_{\beta S-S} \bigcup_{n=1}^{\infty} B_n'' \subset F''$, and it follows that $C \neq \mathrm{cl} \bigcup_{n=1}^{\infty} A_n'$. Since C is an arbitrary clopen subset of $\beta S-S$ containing $\bigcup_{n=1}^{\infty} A_n'$, $\mathrm{cl}_{\beta S-S} \bigcup_{n=1}^{\infty} A_n'$ is not open in $\beta S-S$. (b) now follows since $\bigcup_{n=1}^{\infty} A_n'$ is a cozero set of $\beta S-S$.

4. Remarks.

1. Since extremally disconnected spaces are basically disconnected, the word "basically" may be replaced anywhere in the statement of the theorem by "extremally" and a true statement will result.

2. If S were not pseudocompact, we could have avoided the use of double primes since the V_n could then be chosen so that $\bigcup_{n=1}^{\infty} V_n$ is closed in S. It can be shown that if $S = \beta N - (\beta M - M)$, where M is any denumerable discrete subspace of $\beta N - N$, then S is a pseudocompact space satisfying our hypothesis.

3. It is shown in [2, Remark 3.2] that $\beta S-S$ is not basically disconnected whenever S is a locally compact realcompact space that is not compact. The space S in the above remark in not realcompact.

4. Since a retract of a basically disconnected space is basically disconnected, $\beta S-S$ is not a retract of βS for every space S satisfying our hypothesis. This is a special case of a result of *W. W. Comfort* [1, Theorem 2.7].

5. It follows from a result of *H. P. Rosenthal* [6] that if S is locally compact and basically disconnected, then $\beta S-S$ contains an uncountable number of nonempty pairwise disjoint clopen sets. Hence conclusion (a) follows from (b) if the continuum hypothesis is assumed.

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