MONOTONE BASES AND ORTHOGONAL SYSTEMS IN FRÉCHET SPACES

By

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(Received April 22, 1971)

1. Introduction and Terminology. It has been an endevour of many a mathematician in the past decade to generalise results on bases in Banach spaces to the more general setting of Fréchet spaces. When these results on generalisations and their proofs are set it terms of semi-norms which generate the topology of the Fréchet space in question, it is then our view that they are easily comparable with their counterparts in Banach spaces. *Arsove* [1] in 1958 initiated such a study and got certain interesting results on bases in a Fréchet space (stated and proved in terms of semi-norms) whose counter parts in a Banach space were already known in terms of the norm of the Banach space. Our aim in this note is similar.

We throughout assume, unless contrary stated, that X over the field K stands for a Fréchet space in the Bourbaki terminology and we may assume without loss of generality that its topology T is generated by a family F of semi-norms on X, such that F is closed under maxima of finite subsets of F.

A sequence $\{x_n\}$ in X is called a base if to each $x \in X$, there corresponds a unique sequence of scalars $\{\alpha_n\}$ such that

$$\sum_{i=1}^{n} \alpha_i x_i \to x \quad \text{in } T, \text{ as } n \to \infty . \tag{1.1}$$

The mappings $x_i': X \to K$, $x_i'(x) = \alpha_i$, where α_i's are as (1.1), are continuous linear functionals ([2], p. 453) and further

$$x_i'(x_j) = \begin{cases} 1, & \text{if } i=j \\ 0, & \text{if } i \neq j \end{cases}. \tag{1.2}$$

Thus the system $\{x_n, x_n'\}$ forms a biorthogonal system. If $\{x_n, x_n'\}$ is a biorthogonal system, where $\{x_n\} \subset X$ and $\{x_n'\} \subset X'$, the topological dual of X, we define then the operators $U_n: X \to X$ by the relations

$$U_n(x) = \sum_{i=1}^{n} x_i'(x) x_i , \quad n \geq 1 , \tag{1.3}$$

and call them expansion operators.

If $A\subset X$, we write $Sp\{A\}$ for the space generated by A. If $\overline{Sp}\{A\}$, the closure of $Sp\{A\}$ equals X, we say that A is total in X. We now recall a result to be used in our work:

Theorem 1.1. *Let $\{x_n\}\subset X$, such that $\overline{Sp}\{x_n\}=X$, and $x_n\neq 0$, for $n=1, 2, \cdots$. Then $\{x_n\}$ is a base in X if and only if for each given $p\in F$, there corresponds a $q\in F$ and a constant $M>0$, such that*

$$p\left(\sum_{i=1}^{m}\alpha_i x_i\right)\leq Mq\left(\sum_{i=1}^{n}\alpha_i x_i\right),$$

for each pair of integers m, n; $m\leq n$ and arbitrary scalars $\alpha_1, \cdots, \alpha_n$.

For proof see [4] and for a completely different proof see [3].

2. For the sake of completeness and demonstration we state the following result whose proof runs in the setting of semi-norms which generate the topology of a given X.

Theorem 2.1. *Let $\{x_n, x_n'\}$ be a biorthogonal system in a given X. Then $\{x_n\}$ is a base for $\overline{Sp}\{x_n\}$ if and only if $\{U_n\}$ is equicontinuous.*

Proof. Sufficiency. By hypothesis, to a given $q\in F$, there exists a $q\in F$ and a constant $M>0$, such that

$$p(U_n(x))\leq Mq(x)\,, \qquad \text{for each} \quad x\in X\,, \quad n\geq 1\,.$$

Let $x\in\overline{Sp}\{x_n\}$ and $\varepsilon>0$. Suppose

$$r=\max(p, q)\,.$$

Then there is a $y\in Sp(x_1, \cdots, x_m)$, such that $r(x-y)<\varepsilon$. Observe that $U_n(y)=y$, for $n\geq m$. Hence

$$p(x-U_n(x))\leq p(x-y)+p(U_n(x-y))\,, \qquad n\geq m\,,$$
$$\Rightarrow p(x-U_n(x))<r(x-y)+Mq(x-y)<\varepsilon+M\varepsilon\,, \qquad n\geq m\,.$$

Since $\varepsilon>0$ and $p\in F$ are arbitrary it follows that

$$U_n(x)\to x \quad \text{in} \quad \overline{Sp}\{x_n\} \quad \text{as} \quad n\to\infty\,.$$

If, however,

$$x=\lim_{n\to\infty}\sum_{i=1}^{n}\alpha_i x_i\,,$$

then

$$\lim_{n\to\infty} \sum_{i=1}^{n} (x_i'(x)-\alpha_i)x_i=0 .$$

But each x_j' is continuous and so from the preceding relation $\alpha_j=x_j'(x)$; $j\geq 1$.

For the necessity, we observe that $U_n(x)\to x$ as $n\to\infty$ in $\overline{Sp}\{x_n\}$. Apply now the well-known result of Banach-Steinhans ([2], p. 464) to conclude the equicontinuity of $\{U_n\}$.

3. Monotone Bases. A base $\{x_n\}$ in X will be called *monotone* if for a given $p\in F$, there exists a $q\in F$, such that

(3.1) $$p(U_n(x))\leq q(U_{n+1}(x)) , \quad \text{for each} \quad x\in X , \quad n\geq 1 .$$

We have then

Theorem 3.1. *A base $\{x_n\}$ in X is monotone if and only if to each $p\in F$, there is a $q\in F$, such that*

(3.2) $$p(U_n(x))\leq q(x) , \quad \text{for each} \quad x\in X , \quad n\geq 1 .$$

Proof. Let (3.2) be satisfied. Suppose $\{x_n\}$ is not monotone. Then for a given $p\in F$, there exists an $x_0\in X$ and some $N\geq 1$, such that

$$p(U_N(x_0))>q(U_{N+1}(x_0)) , \quad \text{for all} \quad q\in F .$$

Now

$$\frac{p(U_N(U_{N+1}(x_0)))}{q(U_{N+1}(x_0))}=\frac{p(U_N(x_0))}{q(U_{N+1}(x_0))}>1$$

and if $y=U_{N+1}(x_0)$, then

$$p(U_N(y))>q(y) , \quad \text{for all} \quad q\in F .$$

This contradicts (3.2) and the one-hand conclusion follows.

To prove the necessity of the theorem, let us first of all observe on account of Banach-Steinhans theorem that $\{U_n\}$ is equicontinuous. Take now $p\in F$. Then there is a $q_1\in F$, such that

(3.3) $$p(U_n(x))\leq Mq_1(x) , \quad \text{for each} \quad x\in X , \quad n\geq 1 .$$

Moreover, $\{x_n\}$ is monotone and so there a $q\in F$, such that

(3.4) $$p(U_n(x))\leq q(U_{n+1}(x)) , \quad \text{for each} \quad x\in X , \quad n\geq 1 .$$

Choose $r=\max(q, q_1)$, then replacing x by $U_{n+1}(x)$ in (3.3) we get

$$p(U_n(x))\leq Mr(U_{n+1}(x)) , \quad \text{each} \quad x\in X , \quad n\geq 1 ,$$

and also from (3.4)

$$p(U_n(x)) \leq r(U_{n+1}(x)), \qquad \text{each} \quad x \in X, \quad n \geq 1.$$

Consequently

$$p(U_n(x)) \leq \min(1, M) r(U_{n+1}(x)), \qquad \text{each} \quad x \in X, \quad n \geq 1,$$

and this completes the proof of the result.

4. Orthogonal Systems. We collect certain definitions in order to proceed smoothly.

Definition 4.1. A sequence $\{x_n\}$ in a locally convex space X forms an orthogonal system if for each $p \in F$ there exists a $q \in F$, such that

$$p\left(\sum_{i=1}^{m} \alpha_i x_i\right) \leq q\left(\sum_{i=1}^{n} \alpha_i x_i\right),$$

for each choice of integers m, n; $m \leq n$ and scalars $\alpha_1, \cdots, \alpha_n$.

Definition 4.2. A sequence in a locally convex space forms a finitely orthogonal system if for each $p \in F$, there is a $q \in F$, such that

$$p\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq q\left(\sum_{i=1}^{n+1} \alpha_i x_i\right),$$

for each $n \geq 1$ and arbitrary scalars $\alpha_1, \cdots, \alpha_{n+1}$.

It is clear that definition 4.1 $\Rightarrow$ definition 4.2.

Definition 4.3. A Fréchet space X will be called M-strictly simple if there exists a countable sequence $\{X_n : n \geq 1\}$ of finite dimensional subspaces of X, such that

(4.1) $$X_n \subset X_{n+1}; \dim(X_n) = n, \qquad n \geq 1;$$

(4.2) $$X = \overline{\bigcup_{n \geq 1} X_n};$$

(4.3) Each X_n is the range of a projection P_n on X;

(4.4) To each $p \in F$, there is a $q \in F$, such that

$$p(P_n(x)) \leq M q(x), \qquad \text{for each} \quad x \in X, \quad n \geq 1.$$

We now prove

Theorem 4.1. *Let X have a total orthogonal system. Then X is 1-strictly simple*

Proof. Since $\{x_n\}$ is total in X, therefore

$$X=\overline{Sp}\{x_n\} ;$$

making use of definition 4.1 and then Theorem 1.1 we conclude that $\{x_n\}$ is a base for X. Moreover if $p \in F$, then there is a $q \in F$, such that

$$p(U_m(x)) \leq q(U_n(x)) , \qquad \text{for each} \quad x \in X , \quad n \geq 1 ;$$

and so in particular

$$p(U_m(x)) \leq q(U_{m+1}(x)) , \qquad m \geq 1 , \quad x \in X .$$

Hence $\{x_n\}$ is a monotone base. Let

$$X_n = Sp\{x_1, \cdots, x_n\} .$$

Then $X_n \subset X_{n+1}$; $\dim(X_n)=n$. Since

$$Sp\{x_n\} = \bigcup_{n \geq 1} X_n ,$$

thus (4.1) and (4.2) are proved. Set now P_n as U_n. Then (4.3) easily follows. To prove (4.4) we proceed as in the second part of the proof of Theorem 3.1. This completes the proof of the theorem.

Lastly we have:

Theorem 4.2. *Let $\{x_n\}$ be a monotone base in X. Then $\{x_n\}$ is a total finitely orthogonal system in X.*

Proof. Since $\{x_n\}$ is a base in X, therefore $\{x_n\}$ is total in X. Also $\{x_n\}$ is monotone, therefore if $p \in F$, then there exists a $q \in F$, such that

$$p\left(\sum_{i=1}^{n} x_i'(x)x_i\right) \leq q\left(\sum_{i=1}^{n+1} x_i'(x)x_i\right) , \qquad \text{for each} \quad x \in X , \quad n \geq 1 .$$

Let $\alpha_1, \cdots, \alpha_{n+1}$ be arbitrary $n+1$ scalars. Put $y=\sum_{i=1}^{n} \alpha_i x_i$ and $x=y+\alpha_{n+1}x_{n+1}$. But then

$$x=\sum_{i=1}^{n+1} x_i'(x)x_i .$$

Therefore

$$\sum_{i=1}^{n} x_i'(x)x_i = \sum_{i=1}^{n} \alpha_i x_i ; \qquad x_{n+1}'(x)=\alpha_{n+1} ,$$

and so the result follows.

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