A METHOD FOR SOLVING NON-HOMOGENEOUS INTEGRAL EQUATIONS WITH NON-SYMMETRIC KERNELS HAVING WEAK SINGULARITIES

By

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(Received April 28, 1971)

1. Introduction. *Kreĭn* [3] gave a new method for solving an integral equation of the second kind by deriving a representation formula for the solution. Recently under the assumption of existence of a unique solution, *Kagiwada, Kalaba* and *Schumitzky* [1] gave a better representation formula for the solution $y(t, x)$ of the integral equation:

$$y(t, x)-\int_0^x k(t, u)y(u, x)du=g(t) \qquad \text{for} \quad 0\leqq t\leqq x\,,$$

where x is sufficiently small, $g(t)$ and $k(t, u)$ are continuous, and $k(t, u)$ is symmetric.

In this work, we consider in a Hilbert space framework a non-homogeneous integral equation of the second kind having a non-symmetric real kernel, which is allowed to have a weak singularity [5, p. 49]. We use the technique due to *Schmidt* [6] to symmetrize the integral operator. We show that this process does not affect the original solution. In the construction of orthogonal elements, we use the Gram-Schmidt process. Because the operator is symmetric, this gives rise to a three term recurrence formula, which in turn leads to a system of formulae of simple algebraic form for the approximate solution. The method used is genuinely constructive since we begin the construction with a known element. As the operator after symmetrization is completely continuous and self-adjoint, it follows from *Karush* [2] that the approximate solution, which is the solution of the corresponding finite dimensional problem, tends to the original solution faster than any geometrical progression.

2. Symmetrization. Consider the Hilbert space $L_2(\Omega)$ of all equivalence classes of Lebesgue square integrable real-valued functions in a bounded open connected set Ω of the n-dimensional Euclidean space. Let $M(p, q)$, where p and q are any two points in Ω, be a non-symmetric real kernel having a weak singularity. Our purpose is to find the solution y in $L_2(\Omega)$ of the equation

$$y(p)-\mu\int_{\Omega} M(p, q)y(q)dq=g(p) ,$$

where μ is a given real constant, and g is a given element in $L_2(\Omega)$. Let us denote this equation by

$$(I-\mu M)y=g , \tag{2.1}$$

where I is the identity operator.

Let $M^*(p, q)=M(q, p)$, and also let M^* be the operator corresponding to $M^*(p, q)$. From (2.1),

$$(M^*-\mu M^*M)y=M^*g .$$

Multiplying this by $-\mu$ and adding to (2.1), we have

$$(I-\mu(M+M^*-\mu M^*M))y=(I-\mu M^*)g \tag{2.2}$$

whose kernel is now symmetric.

Theorem 1. *If* (2.1) *has a unique solution in* $L_2(\Omega)$, *then* $(I-\mu M^*)g$ *is not the zero element of* $L_2(\Omega)$, *and both* (2.1) *and* (2.2) *have the same solution.*

Proof. Let $z=(I-\mu M)y$. Then (2.2) becomes

$$(I-\mu M^*)z=(I-\mu M^*)g . \tag{2.3}$$

Let us assume that $(I-\mu M^*)g$ is the zero element of $L_2(\Omega)$. Since the solution of (2.1) satisfies (2.3), it follows that $(I-\mu M^*)z=0$ has a non-trivial solution $z=g$. As Fredholm theorems hold for linear integral equations with weak singularities [4, pp. 59–65], we have by *Fredholm* theorem that its transpose $(I-\mu M)z=0$ has the same number of non-trivial solutions, contradicting by Fredholm alternative that (2.1) has a unique solution. Thus $(I-\mu M^*)g$ is not the zero element.

To prove (2.1) and (2.2) have the same solution, it is sufficient to show that (2.2) has a unique solution. Now if z is the unique solution of (2.3), then it follows from the assumption that (2.1) has a unique solution that y is the unique solution of (2.2). Therefore it is sufficient to prove that (2.3) has a unique solution. We prove this by contradiction. Let us assume that (2.3) has two distinct solutions z_1 and z_2. Then their difference z_1-z_2 is a non-trivial solution of the homogeneous equation $(I-\mu M^*)z=0$. As before, we obtain a contradiction. Hence (2.1) and (2.2) have the same solution.

3. Approximate solution. Since we are interested in finding the solution of (2.1), it is natural to assume that (2.1) has a unique solution. From the theory of linear integral equations with weak singularities, the operator $K \equiv \mu(M+M^* - \mu M^* M)$ is completely continuous [5, pp. 50-53]. Obviously K is self-adjoint. Let $f \equiv (I - \mu M^*)g$. f is known since μ, g and M^* are known. (2.2) becomes

(3.1) $$(I-K)y=f\,.$$

Let $x_0 = f$, and (,) be the inner product. We use the Gram-Schmidt process to construct the sequence $\{x_0, x_1, x_2, \cdots\}$ of orthogonal elements. It follows from the Principle of Mathematical Induction that

$$x_{k+1} = (K - \alpha_k I)x_k - \beta_{k-1}x_{k-1}$$

where

$$\alpha_k = \frac{(Kx_k, x_k)}{(x_k, x_k)}\,,$$

$$\beta_{k-1} = \frac{(Kx_k, x_{k-1})}{(x_{k-1}, x_{k-1})} = \frac{(x_k, x_k)}{(x_{k-1}, x_{k-1})}\,,$$

$\beta_{-1}=0$, $x_{-1}=0$, and $k=0, 1, 2, \cdots$.

Let us assume that an infinite sequence $\{x_n\}$ can be constructed (since if the contrary held, then the following method would terminate with an exact solution). Also let H_n denote the subspace spanned by $\{x_0, x_1, x_2, \cdots, x_{n-1}\}$, L be the linear manifold generated by $\{x_0, x_1, x_2, \cdots, x_n, \cdots\}$, and H denote L and its ideal elements. It is easy to see that H and its orthogonal complement are invariant for K. Thus K is decomposed into two independent parts: one on H and the other on its orthogonal complement. Since H is closed, it follows that H can be regarded as a Hilbert space. Thus the study of K is reduced to its study on the invariant subspaces.

On the subspace H_n, let us construct a linear operator K_n by

$$x_{k+1} = (K_n - \alpha_k I)x_k - \beta_{k-1}x_{k-1} \quad \text{for} \quad k=0, 1, 2, \cdots, n-2\,,$$
$$0 = (K_n - \alpha_{n-1}I)x_{n-1} - \beta_{n-2}x_{n-2}\,.$$

In the equation

(3.2) $$(I-K_n)y_n = f\,,$$

the domain of definition of $I-K_n$ is the intersection of the domain of definition of I and that of K_n, therefore $y_n \in H_n$ and hence can be written in the form

$$y_n = \sum_{k=0}^{n-1} c_k x_k \tag{3.3}$$

where $c_0, c_1, c_2, \cdots, c_{n-1}$ are constants to be determined. Substituting (3.3) into (3.2) and equating coefficients of the x_k's, we have the c_k's determined by the system of equations:

$$c_0 - c_1\beta_0 - c_0\alpha_0 = 1 ,$$
$$-c_{k-1} + (1-\alpha_k)c_k - \beta_k c_{k+1} = 0 \qquad \text{for} \quad k = 1, 2, 3, \cdots, n-1 ,$$

and $c_n = 0$. Thus y_n can be constructed.

The author would like to express his gratitude to Professor *J. L. Howland* and Professor *G. F. D. Duff* for their suggestions.

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