# ON A CLASS OF NONASSOCIATIVE ALGEBRAS

By

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**1. Introduction.** Let $\mathfrak{A}$ be a nonassociative algebra over a field. For every $x$, $y$ in $\mathfrak{A}$, we put

$$D(x, y)=[L(x), L(y)]+[L(x), R(y)]+[R(x), R(y)] ,$$

where $L(x)$ or $R(x)$ is the left or right multiplication of $\mathfrak{A}$; $y\rightarrow xy$, $y\rightarrow yx$, respectively. Now we define a *more generalized standard algebra* to be a nonassociative algebra $\mathfrak{A}$ in which the following conditions are satisfied;

(1) $$D(x, x)=0 ,$$

(2) $$D(x, yz)+D(y, zx)+D(z, xy)=0 ,$$

(3) $D(x, y)$ is a derivation of $\mathfrak{A}$

for all $x$, $y$, $z$ in $\mathfrak{A}$. Clearly every Lie algebra is a more generalized standard algebra. In **3**, we prove that every generalized standard algebra, defined in [5], satisfies these conditions, and that any more generalized standard algebra is a noncommutative Jordan algebra. Hence, every Jordan or alternative algebra is also a more generalized standard algebra. In **2**, we give a lemma for use in **3**. Using this lemma, we remark that the axioms in the definition of generalized standard algebras in [5] are not independent. In **4**, we prove that $\mathfrak{A}^-$ is a general Lie triple system (resp. Malcev algebra) if $\mathfrak{A}$ is a more generalized standard (resp. generalized standard) algebra, where $\mathfrak{A}^-$ is the algebra obtained by defining a new product $[x, y]=xy-yx$ in the same vector space as $\mathfrak{A}$.

Throughout this paper, we assume that the characteristic of the base field is not 2. And we use the following notations;

$$[x, y]=xy-yx , \quad x\cdot y=\frac{1}{2}(xy+yx) , \quad (x, y, z)=(xy)z-x(yz) ,$$
$$H(x, y, z)=(x, y, z)+(y, z, x)+(z, x, y) .$$

**2. Generalized standard algebras.** In [5], Schafer has defined a *generalized standard algebra* to be a nonassociative algebra $\mathfrak{A}$ in which the following conditions are satisfied;

(3) $$D(x, y) \text{ is a derivation of } \mathfrak{A},$$

(4) $$(x, y, x)=0, \text{ i.e. } \mathfrak{A} \text{ is flexible},$$

(5) $$H(x, y, z)x=H(x, y, xz)\ ,$$

(6) $$(x, w, yz)+(y, w, xz)+(z, w, xy)=[x, (y, z, w)]+(x, y, [w, z])$$

for all $x$, $y$, $z$, $w$ in $\mathfrak{A}$.

**Lemma 1.** *Let $\mathfrak{A}$ be a nonassociative algebra, in which (4) and (6) are satisfied. Then the following identities are also satisfied;*

(7) $$(x, y, [z, w])=-(y, x, [z, w])\ ,$$

(8) $$(x, y, [z, w])=-(x, [z, w], y)\ ,$$

(9) $$[x, (y, z, w)]=[x, (z, w, y)]=[x, (w, y, z)]\ ,$$

(10) $$[x, (y, x, z)]=(x, y, [x, z])\ ,$$

(11) $$(x, y, [x, z])=-(x, [y, x], z)$$

*for all $x$, $y$, $z$, $w$ in $\mathfrak{A}$.*

**Proof.** Already (7) and (8) have been proved in [5] and [3], respectively. On the other hand, (9) and (10) have intrisically appeared in [3]. In fact, the following identities have been given there;

(12) $$[x, (y, z, z)]=0\ ,$$

(13) $$(x, y, [x, z])=[x, (x, z, y)]\ .$$

Hence (9) is a direct result of (4) and the identity obtained by linearization of (12), where (10) is a direct result of (9) and (13). Now it remains only to prove (11). It is known that the identity

(14) $$(xy, z, w)+(x, y, zw)=x(y, z, w)+(x, y, z)w+(x, yz, w)$$

is satisfied for all $x$, $y$, $z$, $w$ in any nonassociative algebra. Interchange $x$ (resp. $y$) and $w$ (resp. $z$) in (14), and summarize to obtain

(15) $$([x, y], z, w)+(x, y, [z, w])=(x, [y, z], w)+[x, (y, z, w)]+[(x, y, z), w]\ .$$

Setting $w=z$ in (15), and using (7) and (12), we have that

$$(x, [y, z], z)+[(x, y, z), z]=0\ ,$$

which is equivalent to

(16) $$(z, [y, z], x)=-[z, (y, z, x)]\ .$$

Hence (10) and (16) imply (11).

Now we are in a position to answer a question left to be open in [5].

**Proposition 1.** *Let* $\mathfrak{A}$ *be a nonassociative algebra in which* (4) *and* (6) *are satisfied. Then* (5) *is always satisfied in* $\mathfrak{A}$.

**Proof.** Using (4) and (14), we see that

$$H(x, y, z)x=(xy, z, x)-x(y, z, x)+(y, z, x)x+(z, x, yx)-(z, xy, x)$$

and

$$H(x, y, xz)=(x, yx, z)+x(y, x, z)-(xy, x, z)+(yx, z, x)+(y, x, zx) \\ -(y, x, z)x+(xz, x, y) .$$

It follows that

$$H(x, y, z)x-H(x, y, xz)=([x, y], z, x)+[(y, z, x), x]+(z, x, [y, x]) \\ +(x, [x, y], z)-[x, (y, x, z)]-(y, x, [z, x]) .$$

Using Lemma 1, we see that the right hand side of this identity vanishes, which completes the proof.

## 3. More generalized standard algebras.

Hereafter we denote by $\sum$ the cyclic sum with respect to the three elements $x$, $y$ and $z$; for example, $\sum (x, w, yz) =(x, w, yz)+(y, w, zx)+(z, w, xy)$ .

**Lemma 2.** *In every generalized standard algebra* $\mathfrak{A}$ *the following identities are satisfied;*

$$\sum (x, w, yz)=[w, (z, y, x)] , \tag{17}$$

$$\sum (x, w, [y, z])=2[w, (z, y, x)] , \tag{18}$$

$$\sum (x, w, y\cdot z)=0 , \tag{19}$$

$$\sum (x\cdot y, z, w)= \sum (x, y\cdot z, w) \tag{20}$$

*for all* $x$, $y$, $z$, $w$ *in* $\mathfrak{A}$.

**Proof.** Linearizing of (10), we have

$$[x, (y, w, z)]+[w, (y, x, z)]=(x, y, [w, z])+(w, y, [x, z]) . \tag{21}$$

By (4), (6), (9) and (21), we see that

$$\begin{aligned}\sum (x, w, yz)&=[x, (y, z, w)]+(x, y, [w, z])+(y, w, [z, x]) \\ &=-[x, (y, w, z)]+(x, y, [w, z])+(w, y, [x, z]) \\ &=[w, (y, x, z)]=[w, (z, y, x)] .\end{aligned}$$

The identities (18) and (19) are direct results of (4), (9) and (17). To verify (20), we rewrite (14) as

$$(xy, z, w)-(x, yz, w)=(x, y, z)w+x(y, z, w)+(zw, y, x)$$
$$=(x, y, z)w+x(y, z, w)+z(w, y, x)+(z, wy, x)$$
$$+(z, w, y)x-(z, w, yx) .$$

It follows that

$$\sum (xy, z, w)-\sum (x, yz, w)=H(x, y, z)w+\sum (z, wy, x)$$
$$+\sum (z, w, y)x-\sum (z, w, yx) .$$

Interchange $x$ and $y$, and summarize to obtain

$$\sum (x\cdot y, z, w)-\sum (x, y\cdot z, w)=-\sum (z, w, x\cdot y) .$$

Hence (19) implies (20).

**Proposition 2.** *Every generalized standard algebra is a more generalized standard algebra.*

**Proof.** Since $(x, y, x)=0$ is equivalent to $D(x, x)=0$, it is sufficient to prove (2). In every flexible algebra, the following identities are satisfied;

$$D(x, y)w=[[x, y], w]+2(w, y, x)+2(y, x, w)-(x, w, y) \tag{22}$$

and

$$\sum [x, yz]=H(z, y, x) . \tag{23}$$

Using these identities with (19) and (20), we see that

$$\sum D(x, y\cdot z)w=\sum [[x, y\cdot z], w]-2\sum (x, y\cdot z, w)+2\sum (x\cdot y, z, w)-\sum (x, w, y\cdot z)=0 ,$$

since $H(z, y, x)=-H(y, z, x)$. Similarly, using (22), (23), (18) and Lemma 1, we see that

$$\sum D(x, [y, z])w=\sum [[x, [y, z]], w]+2\sum (w, [y, z], x)+2\sum ([y, z], x, w)$$
$$-\sum (x, w, [y, z])$$
$$=2[H(z, y, x), w]+3\sum (x, w, [y, z])$$
$$=6[(z, y, x), w]+6[w, (z, y, x)]=0 .$$

Hence we see that $2\sum D(x, yz)=2\sum D(x, y\cdot z)+\sum D(x, [y, z])=0$, which completes the proof.

**Corollary.** *Let $\mathfrak{A}$ be either a Jordan algebra or an alternative algebra. Then $\mathfrak{A}$ is a more generalized standard algebra.*

**Proposition 3.** *Every more generalized standard algebra over a field of* *characteristic prime to 6 is a noncommutative Jordan algebra.*

**Proof.** Clearly (1) implies the flexibility of an algebra. Putting $x=y=z$ in (2), we have $D(x^2, x)=0$, equivalently

$$2(w, x, x^2)+2(x, x^2, w)=(x^2, w, x) \tag{24}$$

by (22). On the other hand, we have that

$$(w, x, x^2)+(x, x^2, w)=(x, w, x^2) . \tag{25}$$

In fact, using (14) and the flexibility, we see that $(w, x, x^2)+(x, x^2, w)=(w, x, x^2) -(w, x^2, x) = (w, x, x)x-(wx, x, x) = (x, x, wx)-(x, x, w)x = -(x^2, w, x) = (x, w, x^2)$. Now (24) and (25) imply $(x^2, w, x)=0$, which completes the proof.

A nonassociative algebra $\mathfrak{A}$ is called to be *simple* if it has no non-trivial ideal and $A^2\neq\{0\}$.

**Proposition 4.** *Every simple anti-commutative more generalized standard algebra is a Lie algebra.*

**Proof.** Since $D(x, y)=[L(x), L(y)]$ is a derivation, we have $[[L(x), L(y)], L(z)] =L((x, z, y))$, hence $L(H(x, z, y))=0$. This implies

$$H(x, z, y)=0 . \tag{26}$$

In fact, since $\mathfrak{B}=\{x\in\mathfrak{A}|L(x)=0\}$ is an ideal, it must be $\{0\}$ by the assumption of simplicity. By anti-commutativity, (26) implies the Jacobi's identity.

Now assume that $\mathfrak{A}$ is a not anti-commuatative simple algebra of finite dimension. In case the characteristic of base field is 0, it is known that the algebra $\mathfrak{A}^+$, in which multiplication is defined by $x\cdot y$, is a simple Jordan algebra (see Block [2]). Hence $\mathfrak{A}$ is a Jordan, quasi-associative or quadratic algebra (see Albert [1]).

**4. Algebras $\mathfrak{A}^-$.** In [6], Yamaguti has defined a general Lie triple system as a generalization of a Lie triple system, used in differential geometry and Jordan algebras. Now, we give an equivalent definition for our convenience. A *general Lie triple system* is an anti-commutative algebra $\mathfrak{A}$ with a bilinear mapping; $(x, y)\rightarrow T(x, y)$, of $\mathfrak{A}\times\mathfrak{A}$ into the derivation algebra of $\mathfrak{A}$, satisfying the following conditions;

(i) $T(x, x)=0,$

(ii) $\sum T(x, y)z= \sum (xy)z,$

(iii) $\sum T(xy, z)=0,$

(iv) $$[T(x,y),\ T(z,w)]=T(T(x,y)z,w)+T(z,\ T(x,y)w)$$

for all $x$, $y$, $z$, $w$ in $\mathfrak{A}$.

Now let $\mathfrak{A}$ be a more generalized standard algebra. Putting $T(x,y)=2D(y,x)$, we can easily prove the following

**Proposition 5.** *Let $\mathfrak{A}$ be a more generalized standard algebra. Then, the algebra $\mathfrak{A}^-$ is a general Lie triple system.*

An anti-commutative algebra $\mathfrak{A}$ is called to be a *Malcev algebra* if the following identity is satisfied;

$$(xy)(xz)=((xy)z)x+((yz)x)x+((zx)x)y \tag{27}$$

for all $x$, $y$, $z$ in $\mathfrak{A}$. We note that (27) is equivalent to

$$(x,y,z)x=(y,x,zx)\ . \tag{28}$$

**Proposition 6.** *Let $\mathfrak{A}$ be a generalized standard algebra. Then the algebra $\mathfrak{A}^-$ is a Malcev algebra.*

**Proof.** By $(x,y,z)^-$, we denote the associator of $x$, $y$, $z$ in $\mathfrak{A}^-$; i.e. $(x,y,z)^-=[[x,y],z]-[x,[y,z]]$. Then we see that $(x,y,z)^-=2H(x,y,z)+[y,[z,x]]$ in any flexible algebra. It follows that $\mathfrak{A}^-$ is a Malcev algebra if and only if the identity $[H(x,y,z),x]=H(y,x,[z,x])$ is satisfied for all $x,y,z$ in $\mathfrak{A}$. In a generalized standard algebra, it follows from Lemma 1 that $[H(x,y,z),x]-H(y,x,[z,x])=3[(z,x,y),x]-3(y,x,[z,x])=3[x,(y,x,z)]-3(x,y,[x,z])=0$. Hence $\mathfrak{A}^-$ is a Malcev algebra.

Let $\mathfrak{A}$ be a more generalized standard algebra. By $\mathscr{D}$, we denote the subspace spanned by all $D(x,y)$ in the derivation algebra $\mathscr{D}(\mathfrak{A})$ of $\mathfrak{A}$. The space $\mathscr{D}$ is a subalgebra of $\mathscr{D}(\mathfrak{A})$, hence of the derivation algebra $\mathscr{D}(\mathfrak{A}^-)$ of $\mathfrak{A}^-$. Then the direct sum $\mathscr{L}=\mathfrak{A}+\mathscr{D}$ forms a Lie algebra with respect to the new blacket operation $[\ ,\ ]^*$ defined as follows;

$$[x+D,y+E]^*=[x,y]+D(y)-E(x)+[D,E]+2D(y,x)$$

for all $x$, $y$ in $\mathfrak{A}$ and all $D$, $E$ in $\mathscr{D}$. Although it is not difficult to verify directly the Jacobi's identity, it is also obtained from a theorem for general Lie triple systems (see [6]). This implies

**Proposition 7.** *Every more generalized standard algebra is a reductive Lie admissible algebra in the sence of Sagle* [4].

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