SEQUENTIAL CONDITIONS FOR FIXED AND PERIODIC POINTS

By

Jack Bryant, L. F. Guseman, Jr. and B. Charles Peters

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Many known results on the existence of fixed or periodic points of self-maps of a metric space are simple consequences of the following lemma:

Lemma. *Let* (X, d) *be a metric space and let* $f: X \to X$ *be a continuous mapping, and consider* $\phi_n(x, y) = d(f^n(x), f^n(y))$. *If*

(1) *there exist* x_0, y_0 *with* $f^{n_i}(x_0) \to u$, $f^{n_i}(y_0) \to v$, *and*

(2) $L = \lim_n \phi_n(x_0, y_0)$ *exists,*

then $\phi_m(u, v) = L$ *for each* $m \geq 0$.

Proof. For each $m \geq 0$, f^m is continuous; also $\lim_i \phi_{n_i+m}(x_0, y_0) = L$. Hence

$$L = \lim_i d(f^{n_i+m}(x_0), f^{n_i+m}(y_0)) = d(f^m(u), f^m(v)) .$$

We now give simple proofs of two theorems of *Edelstein*.

Theorem [*1, p. 74*]. *If* $f: X \to X$ *is contractive* $(0 < d(x, y) \Longrightarrow d(f(x), f(y)) < d(x, y))$ *and if there exists* x_0 *with* $f^{n_i}(x_0) \to u$, *then* u *is the unique fixed point of* f.

Proof. Since f is contractive, $\{\phi_n(x_0, f(x_0))\}$ is nonincreasing, so that $L = \lim_n \phi_n(x_0, f(x_0))$ exists. By continuity of f, (1) is satisfied for x_0 and $y_0 = f(x_0)$. Hence for each $m \geq 0$, $d(f^m(u), f^{m+1}(u)) = L$. If u is not a fixed point, then $L = d(f(u), f^2(u)) < d(u, f(u)) = L$. The unicity is clear.

Theorem [*1, p. 76*]. *If* $f: X \to X$ *is* ε*-contractive* $(0 < d(x, y) < \varepsilon \Longrightarrow d(f(x), f(y)) < d(x, y))$ *and if* $f^{n_i}(x_0) \to u$, *then* u *is a periodic point of* f.

Proof. Choose N such that $d(f^{n_N}(x_0), f^{n_{N+1}}(x_0)) < \varepsilon$ and let $k = n_{N+1} - n_N$. For $l \geq n_N$, $\phi_{l+1}(x_0, f^k(x_0)) \geq \phi_l(x_0, f^k(x_0))$ since f is ε-contractive. Hence $L = \lim_l \phi_l(x_0, f^k(x_0))$ exists and $L < \varepsilon$. By continuity of f, (1) is satisfied for x_0 and $y_0 = f^k(x_0)$. Hence for each $m \geq 0$, $d(f^m(u), f^{m+k}(u)) = L < \varepsilon$. If $f^k(u) \neq u$, then $L = d(f(u), f^{k+1}(u)) < d(u, f^k(u)) = L$, a contradiction.

REFERENCES

[1] M. Edelstein, *On fixed and periodic points under contractive mappings*, J. London Math. Soc. **37** (1962), 74–79.

Department of Mathematics
Texas A & M University
College of Science
College Station, Texas 77843
U.S.A.