A NOTE ON WEAK CONTINUITY AND ALMOST CONTINUITY

By

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§1. Introduction: An almost continuous mapping has been defined differently by *Stallings* [10], *Frolik* [3], *Hussain* [4] and *Singal* and *Singal* [9]. *Long* and *Mcgehee* JR. [8] proved that neither of the two almost continuous mappings in the sense of Hussain and in the sense of Stallings implies the the other. *Deb* [1] has also shown by examples that all the four definitions of almost continuity are independent of each other. In an earlier note [7] we proved the following theorem:

If $f: X \to Y$ is one to one, almost open and almost continuous in the sense of Singal, then it is almost continuous in the sense of Hussain.

In the present note we generalize the above theorem by replacing almost continuity in the sense of Singal by weak continuity. It may be mentioned that weak continuity and almost continuity in the sense of Hussain are also independent of each other. In §3 a number of corollaries to the main theorem are given.

We make use of the following definitions:

Definition 1 [*Hussain*, 4]. The function $f: X \to Y$ is almost continuous at $x_0 \in X$ if for each open $V \subset Y$ containing $f(x_0)$, $\overline{[f^{-1}[V]]}$ is a neighborhood of x_0. If f is almost continuous at each point of X, then f is called almost continuous.

Definition 2 [*Singal*, 9]. A mapping $f: X \to Y$ is said to be almost continuous at a point $x \in X$ if for every neighborhood M of $f(x)$, there is a neighborhood N of x such that

$$f(N) \subset \overset{\circ}{\bar{M}} .$$

Definition 3 [2]. A mapping $f: X \to Y$ is said to be $\mathcal{O}$-continuous if for each point $x \in X$ and each neighborhood V of $f(x)$, there is a neighborhood U of x such that $f(\bar{U}) \subset \bar{V}$.

Definition 4 [5]. A mapping $f: X \to Y$ is said to be weakly continuous if for each point $x \in X$ and each neighborhood V of $f(x)$, there exists a neighborhood U of x such that

$$f(U) \subset \bar{V} \; .$$

Remark: Every $\mathscr{O}$-continuous mapping is weakly continuous [9, Remark 3.3] and every almost continuous mapping in the sense of Singal is $\mathscr{O}$-continuous [1]. The following example shows that almost continuity of Hussain does not necessarily imply weak continuity. Weak continuity does not imply almost continuity of Hussain follows from the corresponding result with weak continuity replaced by almost continuity of Singal.

Example [1, p. 89]. Let X be the set of all real numbers with the usual topology and let $f: X \to X$ be the mapping defined by $f(x) = x$ if x is rational and $f(x) = -x$ if x is irrational. Then f is almost continuous in the sense of Hussain but not weakly continuous.

Definition 5 [*Wilansky*, 11]. If $f: X \to Y$ is one to one, then it is almost open iff for every open set $G \subset Y$, $f^{-1}(\bar{G}) \subset [\overline{f^{-1}(G)}]$.

Every open map is almost open.

Definition 6 [*Singal*, 9]. A mapping $f: X \to Y$ is said to be almost open if the image of every regularly open subset of X is an open subset of Y.

Every open map is almost open in the sense of Singal [9].

Definition 7 [6]. A set V is said to be semi-open if ther exists an open set O such that $O \subset V \subset \bar{O}$.

Obviously every open set is semi open.

Throughout the remainder of this paper almost open will refer to almost open in the sense of Wilansky.

§2. Theorem: If $f: X \to Y$ is weakly continuous and for every open set $G \subset Y$, $f^{-1}[\bar{G}] \subset [\overline{f^{-1}(G)}]$ then it is almost continuous in the sense of Hussain.

Proof: Consider $f: X \to Y$. Let $x_0 \in X$. Let V be an open neighborhood of $f(x_0)$ in Y. Then, by weak continuity, there exists an open neighborhood U of x_0 such that $f(U) \subset \bar{V}$. By hypothesis $U \subset f^{-1}(f(U)) \subset f^{-1}[\bar{V}] \subset [\overline{f^{-1}[V]}]$. Thus $x_0 \in U \subset [\overline{f^{-1}[V]}]$. Therefore $[\overline{f^{-1}[V]}]$ is a neighborhood of x_0. Since x_0 is arbitrary it follows that f is almost continuous in the sense of Hussain.

§3. Corollaries:

Corollary 1. If $f: X \to Y$ is one to one, almost open and weakly continuous

in the sense of Hussain.

Proof: By Definition 5.

Corollary 2. If $f: X \to Y$ is one to one, then it is almost continuous in the sense of Hussain if $f^{-1}(\bar{G}) = \overline{[f^{-1}(G)]}$ for every semi open set $G \subset Y$.

Proof: It follows immediately from Definition 5 and the Theorem [1, page 15.: A mapping $f: X \to Y$ is almost continuous in the sense of Singal iff $\overline{[f^{-1}[V]]} \subset f^{-1}[\bar{V}]$ for all semi open subsets V of Y.]

Corollary 3. If $f: X \to Y$ is one to one, $\mathscr{O}$-continuous almost open, then it is almost continuous in the sense of Hussain.

Proof: Follows from the fact that $\mathscr{O}$-continuity implies weak continuity.

For almost continuous mappings in the sense of Singal a result parallel to the above corollary due to Deb [1 page 27] is given below:

If $f: X \to Y$ is $\mathscr{O}$-continuous and almost open in the sense of Singal, then f is almost continuous in the sense of Singal.

Corollary 4. If $f: X \to Y$ is one to one and open, then weak continuity is a sufficient condition for the function f to be almost continuous both in the sense of Hussain and in the sense of Singal.

Proof: We get the result by using Theorem 2.3 [9] and the fact that every open map is almost open.

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