ON THE MEANS OF AN ENTIRE FUNCTION OF SEVERAL COMPLEX VARIABLES

By

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1. Introduction

Let**

$$f(z_1, z_2)=\sum_{m,n=0}^{\infty} a_{m,n} z_1^m z_2^n ,$$

be an entire function of two complex variables. Following Dzrbasyan [1], the orders ρ_1 and ρ_2 of $f(z_1, z_2)$ with respect to the variables z_1 and z_2, respectively, are defined as:

$$\overline{\lim_{r_2\to\infty}}\ \overline{\lim_{r_1\to\infty}} \frac{\log\log M(r_1, r_2)}{\log r_1}=\rho_1 , \quad 0\leq\rho_1\leq\infty , \tag{1.1}$$

and

$$\overline{\lim_{r_1\to\infty}}\ \overline{\lim_{r_2\to\infty}} \frac{\log\log M(r_1, r_2)}{\log r_2}=\rho_2 , \quad 0\leq\rho_2\leq\infty , \tag{1.2}$$

where

$$M(r_1, r_2)= \sup\{|f(z_1, z_2)| : |z_1|\leq r_1,\ |z_2|\leq r_2\} .$$

Define:

$$I_\delta(r_1, r_2)\equiv I_\delta(r_1, r_2; f)=\left\{\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|f(r_1e^{i\theta_1}, r_2e^{i\theta_2})\right|^\delta d\theta_1 d\theta_2\right\}^{1/\delta} , \tag{1.3}$$

and

$$I_\delta^{(i)}(r_1, r_2)=I_\delta\left(r_1, r_2; \frac{\partial f}{\partial z_i}\right), \quad (i=1, 2), \quad \text{where} \quad 1\leq\delta\leq\infty .$$

In this note I prove the theorem:

Theorem: *If $f(z_1, z_2)$ is an entire function of orders ρ_1 and ρ_2, $0\leq\rho_1, \rho_2\leq\infty$, with respect to the variables z_1 and z_2, respectively, then*

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** For simplicity we consider only two variables, though the results can easily be extended to several variables.

(1.3)
$$\overline{\lim_{r_2\to\infty}}\ \overline{\lim_{r_1\to\infty}}\ \frac{\log\left\{r_1\dfrac{I_\delta^{(1)}(r_1, r_2)}{I_\delta(r_1, r_2)}\right\}}{\log r_1}=\rho_1\ ,$$

(1.4)
$$\overline{\lim_{r_1\to\infty}}\ \overline{\lim_{r_2\to\infty}}\ \frac{\log\left\{r_2\dfrac{I_\delta^{(2)}(r_1, r_2)}{I_\delta(r_1, r_2)}\right\}}{\log r_2}=\rho_2\ .$$

2. Lemmas:

It is sufficient to prove (1.3), and the proof of (1.4) will follow similarly. The proof of the same is based on the following lemmas:

Lemma 1: *For any fixed* $r_2\geq 0$, *and* $|z_1|\leq r_1<R_1$,

$$I_\delta^{(1)}(r_1, r_2)\leq (R_1-r_1)^{-1}I_\delta(R_1, r_2)\ .$$

Proof: For a fixed z_2 in the open disk $|z_2|<r_2$, the function $f(z_1, z_2)$ is analytic in the disk $|z_1|<R_1$. Thus applying Cauchy's integral formula to it for the derivative with respect to z_1, we obtain

$$\{I_\delta^{(1)}(r_1, r_2)\}^\delta=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{\partial}{\partial z_1}f(r_1e^{i\theta_1}, r_2e^{i\theta_2})\right|^\delta d\theta_1 d\theta_2\ ,$$
$$=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left|\frac{1}{2\pi i}\int_C\frac{f(\omega_1, z_2)}{(\omega_1-r_1e^{i\theta_1})^2}d\omega_1\right|^\delta d\theta_1 d\theta_2\ ,$$

where C is the disk: $\{\omega_1\colon |\omega_1-r_1e^{i\theta_1}|=R_1-r_1\}$. Putting

$$\omega_1=r_1e^{i\theta_1}+(R_1-r_1)e^{i(\theta_1+\phi_1)}\ ,\quad |f(\omega_1, z_2)|=F(r_1, \theta_1, \phi_1;\ r_2, \theta_2)\equiv F\ ,$$

in this we get

$$\{I_\delta^{(1)}(r_1, r_2)\}^\delta\leq\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}\left\{\frac{1}{2\pi}\cdot\frac{1}{R_1-r_1}\int_0^{2\pi}F(r_1, \theta_1, \phi_1;\ r_2, \theta_2)d\phi_1\right\}^\delta d\theta_1 d\theta_2$$
$$\leq\frac{1}{(R_1-r_1)^\delta}\cdot\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}d\theta_1 d\theta_2\frac{1}{2\pi}\int_0^{2\pi}F^\delta d\phi_1\ ,$$

on applying Holder's inequality. As the integrand is continuous in θ_1, θ_2 and ϕ_1 the order of integration can be changed. Therefore

$$\{I_\delta^{(1)}(r_1, r_2)\}^\delta\leq\frac{1}{(R_1-r_1)^\delta}\cdot\frac{1}{2\pi}\int_0^{2\pi}d\phi_1\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_0^{2\pi}F^\delta d\theta_1 d\theta_2$$

(2.1)
$$=\frac{1}{(R_1-r_1)^\delta}\cdot\frac{1}{2\pi}\int_0^{2\pi}I_\delta^\delta(|r_1+(R_1-r_1)e^{i\phi_1}|, r_2)d\phi_1$$
$$\leq\frac{1}{(R_1-r_1)^\delta}\cdot\frac{1}{2\pi}\int_0^{2\pi}I_\delta^\delta(R_1, r_2)d\phi_1\ ,$$

since $I_\delta(x_1, r_2)$ is an increasing function of x_1 for any $r_2\geq 0$ and $R_1>r_1$. The inequality (2.1) may be written as

$$\{I_\delta^{(1)}(r_1, r_2)\}^\delta \le (R_1-r_1)^{-\delta} I_\delta^\delta(R_1, r_2) .$$

This proves the lemma.

Lemma 2: *For* $r_1 \ge r_1^0 \ge 1$, *any* $r_2 \ge 0$ *and* $\delta \ge 1$,

$$I_\delta^{(1)}(r_1, r_2) \ge \frac{I_\delta(r_1, r_2)}{r_1} \cdot \frac{\log I_\delta(r_1, r_2) - \log I_\delta(r_1^0, r_2)}{\log r_1} .$$

Proof: We have

$$\begin{aligned} I_\delta^{(1)}(r_1, r_2) &= \left\{\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \left|\frac{\partial f}{\partial z_1}(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})\right|^\delta d\theta_1 d\theta_2\right\}^{1/\delta} \\ &= \left\{\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \left|\lim_{\varepsilon\to 0} \frac{f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2}) - f(\overline{r_1 - r_1\varepsilon}\, e^{i\theta_1}, r_2 e^{i\theta_2})}{r_1 \varepsilon e^{i\theta_1}}\right|^\delta d\theta_1 d\theta_2\right\}^{1/\delta} \\ &\ge \lim_{\varepsilon\to 0} \left\{\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} \left(\frac{|f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})| - |f(\overline{r_1 - r_1\varepsilon}\, e^{i\theta_1}, r_2 e^{i\theta_2})|}{r_1 \varepsilon}\right)^\delta d\theta_1 d\theta_2\right\}^{1/\delta} \\ &\ge \lim_{\varepsilon\to 0} \frac{1}{r_1\varepsilon}\left[\left\{\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} |f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})|^\delta d\theta_1 d\theta_2\right\}^{1/\delta}\right. \\ &\quad \left. - \left\{\left(\frac{1}{2\pi}\right)^2 \int_0^{2\pi}\int_0^{2\pi} |f(\overline{r_1 - r_1\varepsilon}\, e^{i\theta_1}, r_2 e^{i\theta_2})|^\delta d\theta_1 d\theta_2\right\}^{1/\delta}\right], \end{aligned}$$

by using Minkowski's inequality ([3], p. 384). Hence

$$\begin{aligned} I_\delta^{(1)}(r_1, r_2) &\ge \lim_{\varepsilon\to 0} \frac{I_\delta(r_1, r_2) - I_\delta(r_1 - r_1\varepsilon, r_2)}{r_1\varepsilon} \\ &= \lim_{\varepsilon\to 0} \frac{I_\delta(r_1 - r_1\varepsilon, r_2)}{r_1\varepsilon}\left\{\frac{I_\delta(r_1, r_2)}{I_\delta(r_1 - r_1\varepsilon, r_2)} - 1\right\} . \end{aligned} \tag{2.2}$$

Further, since $\log I_\delta(r_1, r_2)$ is a convex function of $\log r_1$, for any $r_2 \ge 0$,

$$\log I_\delta(r_1, r_2) = \log I_\delta(r_1^0, r_2) + \int_{r_1^0}^{r_1} \frac{\omega(x_1, r_2)}{x_1} dx_1 , \quad (r_1 \ge r_1^0) \tag{2.3}$$

where $\omega(x_1, r_2)$ increases with x_1 for any $r_2 \ge 0$. From (2.3), we get

$$I_\delta(r_1, r_2) \ge I_\delta(r_1 - r_1\varepsilon, r_2)(1-\varepsilon)^{-\omega(r_1 - \varepsilon r_1, r_2)} = I_\delta(r_1 - r_1\varepsilon, r_2)\{1 + \varepsilon\omega(r_1 - r_1\varepsilon, r_2) + O(\varepsilon^2)\} ,$$

which, when used in (2.2), gives

$$I_\delta^{(1)}(r_1, r_2) \ge \lim_{\varepsilon\to 0} \frac{I_\delta(r_1 - r_1\varepsilon, r_2)}{r_1\varepsilon}\{\varepsilon\omega(r_1 - r_1\varepsilon, r_2) + O(\varepsilon^2)\} = \frac{I_\delta(r_1, r_2)\omega(r_1, r_2)}{r_1} . \tag{2.4}$$

Also, from (2.3), one has

$$\omega(r_1, r_2) \ge \frac{\log I_\delta(r_1, r_2) - \log I_\delta(r_1^0, r_2)}{\log r_1 - \log r_1^0} \ge \frac{\log I_\delta(r_1, r_2) - \log I_\delta(r_1^0, r_2)}{\log r_1} , \tag{2.5}$$

for $r_1 \geq r_1^0$ and $r_2 \geq 0$, since $r_1^0 \geq 1$.

Making use of (2.5) in (2.4), the lemma follows.

Lemma 3: *If $f(z_1, z_2)$ is an entire function of order ρ_1 $(0 \leq \rho_1 \leq \infty)$ with respect to the variable z_1, then*

$$\overline{\lim_{r_2 \to \infty}}\ \overline{\lim_{r_1 \to \infty}} \frac{\log \log I_\delta(r_1, r_2)}{\log r_1} = \rho_1 , \quad 0 < \delta < \infty .$$

Proof: The lemma easily follows, by putting $R_1 = 2r_1$ and taking the limits, from (1.1) and the following inequalities: For $0 \leq r_1 \leq R_1$, $r_2 \geq 0$,

$$I_\delta(r_1, r_2) \leq M(r_1, r_2) \leq \left(\frac{R_1 + r_1}{R_1 - r_1} \right)^{1/\delta} I_\delta(R_1, r_2) .$$

The left hand inequality is obtained from (1.3) whereas the right hand inequality is obtained by applying lemma 1 in [2], for a fixed z_2, to the function $f(r_1 e^{i\theta_1}, r_2 e^{i\theta_2})$.

3. Proof of the main theorem:

From lemma 1 together with (2.3), we get

$$\log I_\delta^{(1)}(r_1, r_2) < \log I_\delta(r_1, r_2) + \log \frac{1}{R_1 - r_1} + \int_{r_1}^{R_1} \frac{\omega(x_1, r_2)}{x_1} dx_1 . \tag{3.1}$$

Choose R_1 such that

$$\frac{\omega(R_1, r_2)}{R_1} = \frac{1}{R_1 - r_1} , \tag{3.2}$$

i.e.

$$R_1 = \frac{r_1}{1 - (\omega(R_1, r_2))^{-1}} \leq \frac{r_1}{1 - (\omega(r_1, r_2))^{-1}} = r_1(1 + o(1)) , \tag{3.3}$$

where $o(1) \to 0$ as $r_1 \to \infty$.

From (3.1), (3.2) and (3.3), we obtain

$$\begin{aligned} \log I_\delta^{(1)}(r, r_2) &< \log I_\delta(r_1, r_2) + \log \frac{\omega(R_1, r_2)}{R_1} + \omega(R_1, r_2) \log \frac{R_1}{r_1} \\ &= \log I_\delta(r_1, r_2) + \log \frac{\omega(R_1, r_2)}{R_1} - \omega(R_1, r_2) \log \left(1 - \frac{R_1 - r_1}{R_1}\right) \\ &= \log I_\delta(r_1, r_2) + \log \frac{\omega(R_1, r_2)}{R_1} - \omega(R_1, r_2) \log \left\{1 - \frac{1}{\omega(R_1, r_2)}\right\} \\ &= \log I_\delta(r_1, r_2) + \log \frac{\omega(R_1, r_2)}{R_1} + 1 + o(1) , \end{aligned}$$

which implies

$$\log\left\{r_1\frac{I_\delta^{(1)}(r_1, r_2)}{I_\delta(r_1, r_2)}\right\} < \log\omega(R_1, r_2) + \log\frac{r_1}{R_1} + 1 + o(1) < \log\omega(R_1, r_2) + 1 + o(1) . \tag{3.4}$$

But, with the help of lemma 3 and (2.3), we can easily prove that

$$\log\omega(R_1, r_2) < (\rho_1+\varepsilon)\log R_1 + O(1) < (\rho_1+\varepsilon)\log r_1 + O(1) + o(1) \quad \text{(by (3.3))} ,$$

for all $r_1 \geq r_1^0 = r_1^0(\varepsilon, r_2)$ and $r_2 \geq 0$, where $O(1)$ is independent of r_1. So using (3.5) in (3.4), we find that

$$\lim_{r_2\to\infty}\lim_{r_1\to\infty}\frac{\log\left\{r_1\dfrac{I_\delta^{(1)}(r_1, r_2)}{I_\delta(r_1, r_2)}\right\}}{\log r_1} \leq \rho_1 .$$

The reverse inequality can easily be obtained from lemma 2.

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