A NOTE ON MALMQUIST'S THEOREM ON FIRST-ORDER DIFFERENTIAL EQUATIONS

By

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§ 1. Introduction

In 1913, *J. Malmquist* [3] proved the following important theorem:

Theorem A. *If $R(z, y(z))$ is a rational function of z and $y(z)$ and if the equation*

$$y'(z)=R(z, y(z)) \;, \tag{1.1}$$

has a solution $y(z)$ single-valued in its domain of existence, then either $y(z)$ is a rational function or (1.1) *is a Riccati equation (Equation* (1.1) *is said to be of Riccati's type if and only if $R(z. y(z))$ is a polynomial in $y(z)$ of degree ≤ 2).*

Later a proof based on *R. Nevanlinna's* theory of meromorphic functions was given by *K. Yosida* [7]. In particular, *Yosida's* argument gives us the following result:

Theorem B. *Let $R(z, y(z))$ be a rational function of z and $y(z)$. Then if the following equation*

$$y'=R(z, y(z)) \;, \tag{1.2}$$

admits a transcendental meromorphic solution $y(z)$ which has only finitely many poles, then $R(z, y(z))$ must be a linear function in $y(z)$, i.e., $R(z, y(z))=r_1(z)+r_2(z)y(z)$ (r_1, r_2 are rational functions).

The argument used in [7] relies heavily on the fact that all the coefficients in $R(z, y(z))$ are rational functions which enables one to use a result of *Valiron* [6]. Unfortunately, there is no corresponding result for a broader class of meromorphic functions. The main purpose of this note, among other things, is to use Nevanlinna theory to extend Theorem B (see Theorem 2 below) by allowing the coefficients in $R(z, y(z))$ to be arbitrary meromorphic functions. The methods developed are different from *Yosida's*. We shall focus our attention of the solutions which grow (in terms of Nevanlinna characteristic function) much faster than all the coefficients in the equation and the number of their poles are required to satisfy certain conditions.)

§ 2. Notation and Preliminary Lemmas

In the sequel, we shall employ the usual notation of Nevanlinna theory. $y(z)$ will always denote a function meromorphic in the whole complex plane. Following Nevanlinna, we define

$$N(r, a)=N(r, a, y)=\int_0^r \frac{[n(t, a)-n(0, a)]}{t} dt+n(0, a)\log r , \tag{2.1}$$

where $n(r, a)=n(r, a, y)$ denotes the number of roots of the equation $y(z)=a$ in $|z|\leq r$.

$$N(r, y)=N(r, \infty, y) .$$

We also define

$$m(r, y)=m(r, \infty, y)=\frac{1}{2\pi}\int_0^{2\pi} \log^+|y(re^{i\theta})|d\theta , \tag{2.2}$$

where $\log^+|x|=\max(\log x, 0)$,

$$m(r, a, f)=m\left(r, \infty, \frac{1}{y-a}\right) \qquad a\neq\infty ,$$

and

$$T(r, y)=m(r, y)+N(r, y) .$$

$T(r, y)$ is clled the characteristic function of y. By virtue of this we are able to give a measure of the growth rate of a meromorphic function.

We shall denote by $S(r, y)$ any quantity satisfying

$$S(r, y)=o\{T(r, y)\} , \tag{2.3}$$

as $r\to\infty$, possibly outside a set of r of finite linear measure.

We shall call a differential polynomial in y and denote by $P_n(y)$ a polynomial of degree at most n in y and its derivatives with the coefficients $a(z)$ satisfying

$$T(r, a(z))=S(r, y) .$$

Lemma 1. (*Milloux* [4]). Let l be a positive integer and

$$\psi(z)=\sum_{i=0}^{l} a_i(z)y^{(i)}(z) \qquad (y^{(0)}\equiv y) , \tag{2.4}$$

where $a_i(z)$ are functions meromorphic in the plane and satisfy

$$T(r, a_i(z))=S(r, y(z)) . \tag{2.5}$$

Then

(2.6) $$m\left(r, \frac{\psi}{y}\right)=S(r, y)\ ,$$

and

(2.7) $$T(r, \psi)\leq(l+1)\,T(r, y)+S(r, y)\ .$$

Remark. If the function y satisfies the condition:

(2.8) $$N(r, y)=S(r, y)\ .$$

then (2.6) and (2.8) yields

(2.9) $$T\left(r, \frac{\psi}{y}\right)=S(r, y)\ .$$

For our estimation, the following result will play a basic role.

Lemma 2. Let $y(z)$ be a transcendental meromorphic function with $N(r, y)=S(r, y)$. Assume that $a_i(z)$ $(i=a, 1, 2, \cdots, l)$ be meromorphic function and satisfy conditions (2.5).
Then

(2.10) $$\begin{aligned}&T(r, y^l+a_1(z)\pi_{l-1}(y)+a_2(z)\pi_{l-2}(y)+\cdots+a_{l-1}(z)\pi_1(y)+a_l(z))\\&=lT(r, y)+S(r, y)\ .\end{aligned}$$

where $\pi_i(y)$ are homogeneous differential polynomial in y of degree i.

Proof.
Set

(2.11) $$p_l(y)=y^l+a_1(z)\pi_{l-1}(y)+\cdots+a_{l-1}(z)\pi_1(y)+a_\pi(z)\ .$$

We rewrite (2.11) as

(2.12) $$\begin{aligned}p_l(y)&=y^l(z)\left(1+\frac{a_1(z)\pi_{l-1}(y)}{y^{l-1}}\cdot\frac{1}{y}+\cdots+\frac{a_{l-1}(z)\pi_1(y)}{y}\cdot\frac{1}{y^{l-1}}+a_l(z)\frac{1}{y^l}\right)\\&=y^l(z)\left(1+\frac{A_1(z)}{y}+\frac{A_2(z)}{y^2}+\cdots+\frac{A_l(z)}{y^l}\right),\end{aligned}$$

where $A_i(z)=a_i(z)\dfrac{\pi_{l-i}(y)}{y^{l-i}}$ $(i=1, 2, \cdots, l)$.

Thus by (2.9) of Lemma 1 and in addition, we have

(2.13) $$T(r, A_i(z))=S(r, y) \qquad (i=1, 2, \cdots, l)\ .$$

Now on the circle $|z|=r$, let

(2.14) $$A(z)=\operatorname*{Max}_{1\leq i\leq l}|A_i(z)|^{1/i} \qquad i=1, 2, \cdots, l\ .$$

Let E_1 be the set of θ in $0\leq\theta\leq 2\pi$ for which $|y(re^{i\theta})|\geq 2A(re^{i\theta})$, and E_2 be the complementary set.

On E_1 we have

$$|p_l(y)|=|y^l(z)|\left|1+\frac{A_1(z)}{y}+\frac{A_2(z)}{y^2}+\cdots+\frac{A_l(z)}{y^l}\right|, \tag{2.15}$$

$$\geq|y^l(z)|\left\{1-\left|\frac{A_1}{y}\right|-\left|\frac{A_2}{y^2}\right|-\cdots-\left|\frac{A_l}{y^l}\right|\right\},$$

$$\geq|y^l(z)|\left\{1-\frac{1}{2}-\frac{1}{2^2}-\cdots-\frac{1}{2^l}\right\},$$

$$=\frac{1}{2^l}|y^l(z)|.$$

Hence, from this, (2.13) and (2.14), we have

$$lm(r,y)=m(r,y^l) \tag{2.16}$$

$$=\frac{1}{2\pi}\int_{E_1}\log^+|y^l(re^{i\theta})|d\theta+\frac{1}{2\pi}\int_{E_2}\log^+|y^l(re^{i\theta})|d\theta,$$

$$\leq\frac{1}{2\pi}\int_0^{2\pi}\log^+|2^l p_l(y)|d\theta+\frac{1}{2\pi}\int_{E_2}\log^+|2A^{(z)}|^l d\theta,$$

$$=m(r,p_l(y))+l\log 2+S(r,y).$$

Adding $lN(r,y)$ on both sides of the above inequality, we obtain

$$lT(r,y)\leq T(r,p_l(y))+lN(r,y)+S(r,y), \tag{2.17}$$

$$=T(r,p_l(y))+S(r,y),$$

Since $N(r,y)=S(r,y)$.

On the other hand, it is easily shown from the expression (2.12) and by induction on l that

$$T(r,p_l(y))\leq lT(r,y)+S(r,y). \tag{2.18}$$

Thus, combining (2.17) and (2.18), we obtain (2.10).

In a polynomial $p(z,y_0,y_1,\cdots,y_n)$, we shall denote $k_0+k_1+\cdots+k_n$ and $k_1+2k_2+\cdots+nk_n$ as the dimension and weight of a term $a(z)y_0^{k_0}y_1^{k_1}y_2^{k_2}\cdots y_n^{k_n}(a(z)\not\equiv 0)$ respectively. The degree of $p(y)$ is the maximal dimension among all its terms. We shall call $p(y)$ a polynomial of degree d as non-degenerate if

$$\sum a(z)\not\equiv 0, \tag{2.19}$$

where $a(z)$ are coefficients in $p(y)$ and the summation is taken over all terms with dimension d and the weight being maximal.

Lemma 3. Let $y(z)$ be meromorphic transcendental with $N(r, y)=S(r, y)$. If $p(y)$ is a non-degenerate differential polynomial of degree d, then

(2.19) $$T(r, p(y))=dT(r, y)+S(r, y) .$$

Proof. By a result of *Y. Tumura* [5] one can express

(2.20) $$y^{(n)}=y(F^n+p_{n-1}(F)) \qquad n=1, 2, \cdots .$$

where $F=\frac{y'}{y}$. We note that $T(r, F)=N(r, F)+m(r, F)=S(r, y)$.

Thus by substituting this into $p(y)$, $p(y)$ will become the form $A(z)y^d+p_{d-1}(y)$. The rest follows by lemma 2.

Lemma 4. (*Clunie* [1]). Suppose that $p_m(y)$ and $Q_n(y)$ be two differential polynomials in y and that

(2.21) $$y^n(z)p_m(y)=Q_n(y) .$$

Then

(2.22) $$m\{r, p_m(y)\}=S(r, y) \qquad \text{as} \quad r\to\infty .$$

§ 3. Main Results

We first prove a general result.

Theorem 1. *Let $R_i(z, y(z), y_1(z), \cdots, y_k(z))$ $i=1, 2$, be given rational functions in $y_0(z), y_1(z), \cdots y_k(z)(y_i(z)\equiv y^{(i)}(z))$ with meromorphic functions as their coefficients.*

Assume that

(3.1) $$R_i(z, y_0(z), y_1(z), \cdots, y_k(z))\equiv\frac{p_{il}(z, y_0, \cdots, y_k)}{q_{il}(z, y_0, \cdots, y_k)} ,$$

where, $p_{il}(z, y_0, y_1, \cdots, y_k)$, $q_{il}(z, y_0, y_1, \cdots, y_k)$ are two relatively prime polynomials in $y_0, y_1, \cdots, y_k$ $(i=1, 2)$ with degree n_i, m_i $(i=1, 2)$ respectively.

Assume that

(3.2) $$p_{21}(z, y_0, \cdots, y_k)=y^{n_2}(z)+p_2(z, y_0, \cdots, y_k) ,$$

and

(3.3) $$q_{21}(z, y_0, \cdots, y_k)=y^{m_2}(z)+q_2(z, y_0, \cdots, y_k) ,$$

where $p_2(z, y_0, \cdots, y_k)$ and $q_2(z, y_0, \cdots, y_k)$ are two polynomials in $y_0, \cdots, y_k$ with degree less than n_2, m_2 respectively.

Further assume that p_{11} and q_{11} both are non-degenerate. Then if the differential equation

$$R_1(z, y_0, \cdots, y_k)=R_2(z, y_0, \cdots, y_k)\ , \tag{3.4}$$

admits a transcendental meromorphic solution $y(z)$ with

$$N(r, y)=S(r, y)\ , \tag{3.5}$$

and such that

$$T(r, a(z))=S(r, y) \qquad \text{as} \quad r\to\infty\ , \tag{3.6}$$

hold for all the coefficients $a(z)$ in R_1 and R_2, we must have

$$n_1+m_1\geqq|n_2-m_2|\ .$$

Proof. Since by assumption that p_{11} and q_{11} both are non-degnerated, and the property that $T(r,f_1f_2)\leqq T(rf_1)+T(r,f_2)$ (see e. g. [2]) we have

$$T(r, R_1)\leqq T(r, p_{11})+T\left(r, \frac{1}{q_{11}}\right).$$

It follows by lemma 1 and Nevanlinna's first fundamental theorem, that

$$\begin{aligned} T(r, R_1)&\leqq n_1T(r, y)+m_1T(r, y)+S(r, y)\ , \\ &=(n_1+m_1)T(r, y)+S(r, y)\ . \end{aligned} \tag{3.7}$$

while

$$\begin{aligned} T(r, R_2)&\geqq T(r, p_{21})-T(r, q_{21}) \\ &=n_2T(r, y)-m_2T(r, y)+S(r, y)\ , \\ &=(n_2-m_2)T(r, y)+S(r, y)\ . \end{aligned} \tag{3.8}$$

Since p_{21} and q_{21} are interchangeable, we thus have

$$T(r, R_2)\geqq|n_2-m_2|\,T(r, y)+S(r, y)\ .$$

Our assertion follows from this, (3.7), and the fact that $T(r, R_1)=T(r, R_2)$.

Now let us consider the special case

$$R_1(z, y_0, y_1, \cdots, y_k)=y' \quad \text{and} \quad R_2(z, y_0, y_1, \cdots, y_k)$$

is a rational function of y only.

Thus equation (3.4) assumes the form

$$y'=\frac{a_1(z)+a_2(z)y+\ \cdots\ +a_{n_2}(z)y^{n_2}}{b_1(z)+b_2(z)y+\ \cdots\ +b_{m_2}(z)y^{m_2}}\ . \tag{3.9}$$

$(a_{n_2}\neq 0,\ b_{m_2}\neq 0)$.

Now suppose that (3.9) has a transcendental meromorphic solution $y(z)$ satisfying

(3.10) $$N(r, y)=S(r, y) ,$$

(3.11) $$T(r, a_i(z))=S(r, y) \qquad (i=1, 2, \cdots, n_2) ,$$

and

(3.12) $$T(r, b_j(z))=S(r, y) \qquad (j=1, 2, \cdots, m_2) .$$

Then by (3.10) and (2.5) of lemma 1 we have

(3.13) $$T(r, y')\leqq m\left(r, \frac{y'}{y}\right)+m(r, y)+N(r, y') ,$$
$$=S(r, y)+T(r, y)+S(r, y) .$$
$$=T(r, y)+S(r, y) .$$

It follows from this and theorem 1 that

(3.14) $$|m_2-n_2|\leqq 1 .$$

Now suppose that $m_2\neq 0$. Then we have (i) $n_2=m_2-1$ or (ii) $n_2=m_2$ or (iii) $n_2=m_2+1$.

In case (i) we have, according to (3.9) that

$$b_{m_2}(z)y^{m_2}y'+b_{m_2-1}(z)y^{m_2-1}y'+\cdots=a_{n_2}(z)y^{n_2}+\cdots .$$

Hence

(3.15) $$y^{m_2}(b_{m_2}y')+p_{m_2-1}(y)=Q_{n_2}(y) ,$$

or

(3.16) $$y^{m_2-1}[b_{m_2}(z)yy'+b_{m_2-1}(z)y']+p_{m_2-2}(y)=Q_{n_2}(y) ,$$

where $Q_{n_2}(y)$ is a differential polynomial in y of degree at most n_2.

Applying lemma 4 to (3.15) and (3.16) we obtain

(3.17) $$m(r, b_{m_2}y')=S(r, y) ,$$

and

(3.18) $$m(r, b_{m_2}y'y-b_{m-1}(z)y')=S(r, y) .$$

By (3.10) we deduce

(3.19) $$T(r, b_{m_2}y')=S(r, y) ,$$

and

(3.20) $$T\{r, y'(b_{m_2}y-b_{m_2-1})\}=S(r, y) .$$

Thus it follows from (3.19) and (3.20) that

$$T(r, b_{m_2}y-b_{m_2-1})\leqq T\{(r, y'(b_{m_2}y-b_{m_2-1})\}+T\left(r, \frac{1}{b_{m_2}y'}\right), \tag{3.21}$$

$$=S(r, y) .$$

This gives a contradiction, since

$$T(r, b_{m_2}y-b_{m_2-1})=T(r, y)+S(r, y) . \tag{3.22}$$

Case (ii) and case (iii) can be handled in a similiar manner and will lead to the same contradiction.

Thus we must have $m_2=0$.

Now according to (3.14) we conclude $n_2=1$.

Thus the following theorem is proved.

Theorem 2. *If the differential equation*

$$y'=\frac{a_1(z)+a_2(z)y+\cdots+a_{n_2}(z)y^{n_2}}{b_1(z)+b_2(z)y+\cdots+b_{m_2}(z)y^{m_2}}, \tag{3.23}$$

has a transcendental meromorphic solution $y(z)$ *such that conditions* (3.10), (3.11) *and* (3.12) *are satisfied. Then it is necessary that the right hand side of equation* (3.23) *is linear in* y.

Pemark. The above argument also shows that the same conclusion holds if one replaces y' by $y^{(n)}$ $(n\geqq 0)$ in the equation (3.23).

The following two results follow immediately from the proof of Theorem 2.

Corollary 1. The following differential equation

$$a_1(z)y(z)y'(z)+a_2(z)y'(z)+a_3(z)\equiv 0 ,$$

with $a_1(z)\not\equiv 0$, $a_2(z)\not\equiv 0$ has no entire solutions $y(z)$ which satisfies $T(r,a_i(z))=S(r, y(z))$. $i=1, 2, 3$.

Corollary 2. The following differential equation

$$a_1(z)y(z)y'(z)+a_2(z)y(z)+a_3(z)\equiv 0 ,$$

with $a_1(z)a_3(z)\not\equiv 0$ has no entire solution $y(z)$ which satisfies

$$T(r, a_i(z))=S(r, y(z)) \qquad i=1, 2, 3 .$$

REFERENCES

[1] J. Clunie, On integral and meromorphic functions, Jour. Lond. Math. Soc. 37 (1962), p. 20.

[2] W.K. Hayman, Meromorphic functions, Oxford, 1964, p. 5.

[3] J. Malmquist, Sur les fonctions a un rompre fini de branches definies par les equations differentielles du premier ordre, Acta Math. 36 (1931), pp. 297–343.

[4] H. Milloux, Les fonctions meromorphes et leurs derivees, Paris (1940).

[5] Y. Tumura, On the extensions of Borel's theorem and Saxer-Csillag's theorem, Proc. Phys-Math. Soc. Jap. III. s. 19 (1937), p. 32.

[6] G. Valiron, Sur la derivee des fonctions algebroides, Bull. de la Soc. Math. de France, (1931), p. 34.

[7] K. Yosida, A generalization of Malmquist's theorem, Jap. J. Math. 9 (1932), pp. 253–256.

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