ON COMPACTIFICATIONS OF TYCHONOFF SPACES

By

T. Thrivikraman

Abstract: βX denotes the Stone-Čech compactification of a Tychonoff space X. Some topological properties of $\beta X - X$ are characterised in terms of lattice-theoretic properties of the upper semi-lattice $K(X)$ of all Hausdorff compactifications of X. Also, we construct the space $\beta X - X$ from $K(X)$ when X is locally compact; when X is not locally compact, the compact subsets of $\beta X - X$ are specified.

§1. Introduction

In [6], the author has established that when X is locally compact, the topological properties of $\beta X - X$ are related contravariantly in some sense, to the lattice theoretic properties of the lattice $K(X)$ of compactifications of X. But the situation is not that nice when X is not locally compact even if we restrict our morphisms to homeomorphisms and 'lattice-isomorphisms'. Examples in the two opposite directions to establish the above statement are given in [5] and [6] respectively. However in section 2, we give some direct results which characterise some topological properties of $\beta X - X$ in terms of lattice-theoretic properties of $K(X)$, the semi-lattice of all Hausdorff compactifications of X. But in general, the problem is naturally not that trivial. It is known that if αX is any compactification of X and R is any equivalence relation on αX which is trivial on X and which is closed in the product space $\alpha X \times \alpha X$, then the quotient space $\alpha X / R$ is a compactification of X. ([1] (129) Ex. E). In particular, all the compactifications of X are given by the closed equivalence relations on βX which are trivial on X. The converse problem arises, viz., can we get back the space $\beta X - X$ from the semi-lattice $K(X)$? In section 3, we get back the space $\beta X - X$ from $K(X)$ when X is locally compact, and the compact sets of $\beta X - X$ otherwise. Theorem 1 of [4] follows as a corollary.

This leads to the more general problem: What are all semi-lattices which are candidates for being $K(X)$ for some space X? A complete characterisation has been given in [3] for locally compact spaces X.

Conventions and notations

In this paper, a space means a Tychonoff space, *i.e.*, a completely regular Hausdorff space.

βX denotes the Stone-Čech compactification of X. $K(X)$ stands for the upper-complete semi-lattice ([1]) of all compactifications of a space X, modulo their usual equivalence, under the usual order.

$\mathfrak{F}\alpha(X)$ denotes the family of partition classes $\beta X - X$ corresponding to the compactification αX of X.

§ 2. The following results will be used in this paper. *Magill, K. D. Jr.* [4] has proved them in the particular case when X is locally compact, but they remain to be true in the general case when X is any arbitrary Tychonoff space. See also [5]. Result 1.1 can be proved by using Ex. E (129) of [1] and the rest routine or on the same lines as in [4].

2.1 Result. Let $\alpha X \in K(X)$; Let $K_1, K_2, \cdots, K_N$ be a finite number of pairwise disjoint non-empty compact subsets of $\alpha X - X$. Let δX be the space obtained by identifying $K_1, K_2, \cdots, K_N$ separately and giving the quotient topology from αX. Then $\delta X \in K(X)$.

Notation: αX is denoted by $\alpha(X; K_1, K_2, \cdots, K_N)$.

2.2 Result. $\alpha(X; K_1) \wedge \alpha(X; K_2) = \begin{vmatrix} \alpha(X; K_1, K_2) \text{ if } K_1 \cap K_2 = \phi, \\ \alpha(X; K_1 \cup K_2) \text{ if } K_1 \cap K_2 \neq \phi. \end{vmatrix}$

$\alpha(X; K_1) \vee \alpha(X; K_2) = \alpha(X; K_1 \cap K_2)$.

2.3 Result. αX is a dual atom in $K(X)$ if and only if there exist distinct points p and q in $\beta X - X$ such that $\alpha X = \alpha(X; \{p, q\})$.

2.4 Definition. A compactification αX of a space X is called a primary compactification if $\mathfrak{F}(\alpha X)$ has precisely one non-singleton.

2.5 Result. αX is a primary compactification of X if and only if $\alpha X \neq \beta X$ and there do not exist two dual atoms δX and $\delta' X$ of $K(X)$ such that $\alpha X \wedge \delta X = \alpha X \wedge \delta' X \neq \alpha X$ and such that the only dual atoms $> \delta X \wedge \delta' X$ are δX and $\delta' X$.

§ 3. In this section, we prove some direct results indicating the relations between topological properties of $\beta X - X$ and lattice properties of $K(X)$.

3.1 Result. $K(X)$ is distributive if and only if $|\beta X - X| < 3$.

Proof: $|\beta X - X| < 3$ if and only if $|K(X)| \leq 2$. So if $|\beta X - X| < 3$, $K(X)$ is

trivially distributive. If $|\beta X - X| \geq 3$, let a, b, c be three distinct elements of $\beta X - X$. Consider the compactifications βX, $\alpha(X; \{a, b\})$, $\alpha(X; \{b, c\})$, $\alpha(X; \{a, c\})$ and $\alpha(X; \{a, b, c\})$. They form a sublattice of $K(X)$ isomorphic to $M_5 = $ and so $K(X)$ is not distributive.

3.2. Result. $K(X)$ is modular if and only if $|\beta X - X| \leq 4$.

Proof: If $|\beta X - X| \leq 4$, then it can be easily checked that $K(X)$ is modular. If $|\beta X - X| > 4$, choose distinct points a, b, c, d, e in $\beta X - X$. The compactifications βX, $\alpha(X; \{a, c\}, \{b, d\})$ $\alpha(X; \{a, b\}, \{d, e\})$, $\alpha(X; \{a, b\}, \{c, d, e\})$ and $\alpha(X; \{a, b, c, d, e\})$ form a sublattice of $K(X)$ isomorphic to the lattice $N_5 = $ and so $K(X)$ is not modular.

However, we have

3.3 Result. The primary compactifications satisfy the modular law.

Proof: Let $\alpha_1 X = \alpha(X; H)$, $\alpha_2 X = \alpha(X; K)$, $\alpha_3 X = \alpha(X; L)$ and let $\alpha_1 X \leq \alpha_3 X$ i.e., $L \subseteq H$. Then

$$\alpha_1 X \vee (\alpha_2 X \wedge \alpha_3 X) = \begin{vmatrix} \alpha_1 X \vee \alpha(X; K \cup L) & \text{if } K \cap L \neq \phi \\ \alpha_1 X \vee \alpha(X; K, L) & \text{if } K \cap L = \phi \end{vmatrix}$$

$$= \begin{vmatrix} \alpha(X; H \cap (K \cup L) & \text{if } K \cap L \neq \phi \\ \alpha(X; H \cap K, H \cap L) & \text{if } K \cap L = \phi \end{vmatrix}$$

$$= \begin{vmatrix} \alpha(X; (H \cap K) \cup L) & \text{if } K \cap L \neq \phi \\ \alpha(X; H \cap K, L) & \text{if } K \cap L = \phi \end{vmatrix} \text{ since } L \subseteq H$$

$$= |\alpha(X; H \cap K) \wedge \alpha(X; L) \text{ since } L \subseteq H$$

$$= |(\alpha_1 X \vee \alpha_2 X) \wedge \alpha_3 X,$$

Hence the result.

3.4 Result. $K(X)$ has a zero element but no atom if and only if $\beta X - X$ is compact connected.

Proof: $K(X)$ has zero if and only if X is locally compact, i.e., if and only if $\beta X - X$ is compact. Further if $K(X)$ has an atom, it can be only a two-point compactification since otherwise let αX be an atom of $K(X)$ which is not a two-point-compactification. Then $\mathfrak{F}(\alpha X)$ has more than two elements. Take the set

union of any two of them and the resulting partition gives a compactification of X smaller than αX and larger than the one-point-compactification which is a contradiction. So if $K(X)$ has an atom, it is a two-point-compactification which gives a partition of $\beta X - X$ into two disjoint closed sets, so that $\beta X - X$ is not connected. Conversely if $\beta X - X$ is not connected, then there exists a Hausorff partition of $\beta X - X$ into two disjoint closed sets which gives a two-point-compactification so that there is an atom. Hence the result.

3.5 Definition. A lattice L is called upper semi-complemented if for every $d \in L$, there exists $d' \in L$ such that $d \vee d' = 1$.

3.6 Result. The complete lattice $K(X)$ is upper semi-complemented if and only if for every Hausdorff quotient K of $\beta X - X$, there exists a quotient K' such that $\beta X - X$ is homeomorphic to a closed subspace of $K \times K'$.

Proof: $K(X)$ is upper semi-complemented if and only if given any closed partition π of $\beta X - X$, there exists a closed partition π' of $\beta X - X$ such that $A \cap B$ is either empty or singleton for evepy $A \in \pi$ and $B \in \pi'$. This happens if and only if the map $x \to (\pi_x, \pi'_x)$ from $\beta X - X$ into $(\beta X - X)/\pi \times (\beta X - X)/\pi'$ is one-one. The map is clearly continuous and its image can be seen to be a closed subset of range space. Now the assertion follows.

Note. $K(N)$ is not upper semi-complemented where N is the countable discrete space. For $[1, \Omega]$ is a quotient of $\beta N - N$ (see [2]) and $[1, \omega]$ is not a closed subspace of $\beta N - N$.

3.7 Result. If $K(X)$ is complemented, then $\beta X - X$ is totally disconnected.

Proof: Let $x, y \in \beta X - X$ be two distinct points. Then $\alpha(X; \{x, y\})$ is a dual atom of $K(X)$. Since $K(X)$ is complemented, there exists a compactification $\alpha' X$ of X such that $\alpha' X \wedge \alpha(X; \{x, y\}) = 0$ and $\alpha' X \vee \alpha(X; \{x, y\}) = 1$. Since $\alpha' X \wedge \alpha(X; \{x, y\}) = 0$, there can be at the most two partition classes in $\mathfrak{F}(\alpha' X)$ in which case one contains x and the other y. But since $\alpha' X \vee \alpha(X; \{x, y\}) = 1$, there should exist two such partition classes. i. e., $\beta X - X = A \cup B$ where $x \in A$, $y \in B$ and A and B are both closed being partition classes corresponding to a Hausdorff compactification of a locally compact space. So $\beta X - X$ is totally disconnected.

Note. Converse is not true; for, $K(N)$ is not even upper semi-complemented.

§ 4. In this section, we get back the space $\beta X - X$ from $K(X)$ when X is locally compact; the compact subsets of $\beta X - X$ otherwise.

Remark. If $|K(X)| \leq 2$, the situation is trivial and is not included in the following discussion.

4.1 Notation. The set of all dual atoms of $K(X)$ will be denoted by D.

4.2. Definition. Two distinct dual atoms of $K(X)$ are said to overlap if there are precisely three dual atoms above their lattice intersection.

4.3 Definition. Let d_1, d_2 be two overlapping dual atoms of $K(X)$. We say that a third dual atom d' is hinged with d_1, d_2 if the following happens:

(i) d' overlaps with d_1 as well as with d_2.

(ii) there are precisely six dual atoms above the lattice intersection of d', d_1 and d_2.

4.4 Definition. Let d_1, d_2 be two overlapping dual atoms of $K(X)$. The set of all dual atoms hinged with d_1 and d_2 will be called the point $|d_1 d_2|$.

4.5 Remark. For the semi-lattice $K(X)$, we notice the following:

(i) Any two distinct dual atoms of $K(X)$ are either overlapping or there exists no other dual atom above their intersection.

(ii) If d_1, d_2 are overlapping dual atoms, then the corresponding identifications in $\beta X - X$ has exactly one common point. In other words, if $d_1 = \alpha(X; \{a, b\})$, $d_2 = \alpha(X; \{c, d\})$, ahen $\{a, b\} \cap \{c, d\}$ is a singleton (say) $a = c$.

(iii) The set of all dual atoms hinged with d_1, d_2 uniquely determines a point of $\beta X - X$, viz., a.

(iv) If $d_3, d_4 \in |d_1 d_2|$ and d_2, d_4 are distinct, then they are overlapping and $|d_3 d_4| = |d_1 d_2|$.

(v) Any two distinct sets $|d_1 d_2|$ and $|d' d''|$ intersect setwise precisely in a singleton.

4.6 Notation. The set of all subsets of D of the form $|d_1 d_2|$ will be denoted by F.

4.7 Definition. Let A be a subset of F with more than one element. Then a dual atom $d \in K(X)$ is said to be determined by A if d occurs as the unique set intersection of two members of A.

4.8 Definition. Let A be a subset of F with more than one element. Let $\mathfrak{D}$ be the collection of all dual atoms determined by A. We say that A is F-compact provided (i) $\bigwedge_{d \in \mathfrak{D}} d$ exists and (ii) $\mathfrak{D} = \mathfrak{D}'$ where $\mathfrak{D}'$ is the collection of all dual atoms $\geq \bigwedge_{d \in \mathfrak{D}} d$ in $K(X)$. We say that a singleton subset of F is F-compact.

4.9 Theorem. There is a bijection from F to $\beta X - X$ which carries F-compact sets to compact sets of $\beta X - X$ and vice versa. Further the complements of F-compact sets of F form a topology for F if and only if X is locally compact. In this case, F is homeomorphic to $\beta X - X$.

Proof: Let $|d_1 d_2| \in F$. Let $d_1 = \alpha(X; \{a, b\})$ and $d_2 = \alpha(X; \{a, c\})$ (see Remark 3.5 (ii)). Then if $d' \in |d_1 d_2|$, then $d' = \alpha(X; \{a, d\})$ for some $d \in \beta X - X$ by 3.5 (iii). Thus we get a natural bijection $\phi \colon F \to \beta X - X$ defined as $\phi(|d_1 d_2|) = a$.

Let A be F-compact. If A is a singleton, then $\phi(A) = a$ for some $a \in \beta X - X$ which is uniquely determined. If A contains more than one point, let $|d_1 d_2|$ and $|d_3 d_4|$ be distinct members of A. Let $|d_1 d_2|$ uniquely determines a and $|d_3 d_4|$ uniquely determine b. Then $\alpha(X; \{a, b\})$ is a dual atom determined by A. Now notice (either by using 1.4 or independently) that $\bigwedge\limits_{d \in \mathfrak{D}} d$ if it exists, can be only a primary compactification, say, $\alpha(X; H)$. Since A is F-compact, the collection of dual atoms $\geq \bigwedge\limits_{d \in \mathfrak{D}} d$ is precisely $\mathfrak{D}$. This proves that $\phi(A)$ is compact in $\beta X - X$ and it is actually H. The converse is trivial.

Further X is locally compact if and only if $\beta X - X$ is compact which happens if and only if F is F-compact since $\phi(F) = \beta X - X$. In this case, defining each F-compact set to be closed, we get a topology for F. This obviously makes ϕ a homeomorphism since ϕ carries F-compact sets to compact sets and vice versa.

4.10 Corollary. Let X and Y be Tychonoff spaces. If $K(X)$ and $K(Y)$ are isomorphic, then there exists a bijection h from $\beta X - X$ onto $\beta Y - Y$ which preserves compact sets in both directions.

In particular,

4.11 Corollary. (*Magill, K.D., Jr.*) Let X and Y be locally compact. If $K(X)$ and $K(Y)$ are isomorphic, then $\beta X - X$ and $\beta Y - Y$ are homeomorphic.

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Madurai University
Madurai-2, INDIA.