GENERALIZED MODELS, PROBABILITY OF FORMULAS, DECISIONS AND STATISTICAL DECIDABILITY OF THEOREMS

By

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(Received Jan. 5, 1971)

The paper is based on my published results, *e.g.* in Japan, and it has a certain expositive character.

The topic belongs to most important and basic ones of Mathematics, Statistics and Computers; so we find here all names of greatest Mathematicians of the history.

I have greatly enjoyed finishing my paper on a subject which has fascinated all scientists of those domains since the origin of science—finishing it in Mathematics Department of Technological University of Delft and therefore it is my pleasure to acknowledge my indebtedness to my all new Colleagues with the Chairmen, Professor Dr. *A. W. Goosens*, Professor Dr. *F. Loonstra*, Professor Dr. *Ir. J. W. Sieben* and Professor Dr. *J.W. Cohen*, who introduced me in my new circle as Visiting Professor with their constant best wishes and cooperations and to Ing. S.J. de Lange for friendly talks in ones.

Working in decidability and decision methods[1] I have given different generalizations of the satisfiability definition receiving generatizations of basic theorems of aforementioned domains; I introduced generalized models with their asymptotic adequateness or adequateness with conditions. My invariance relation is the source of many results of different scientists, *e.g.* it is the origin of ,,forcing",[2] in its syntactical development.

Ones lead to many technical applications. I cite from my note [38] based on my discussion on meetings of the International Mathematical Congress in Nice, 1-10, 9. 1970.

"My truncated generalized model contains forcing models—simply: forcing —and forcing follows from the proof of my invariant relation with many other

[1] The paper is connected with my lectures on seminars of Jerzy Stupecki in 1950-57 years and on meetings of Wroctaw Mathematical Society in those years.

[2] With ,,potential truth".

different assumptions equivalent to my invariant relation, see [3], [5], [23] and p. 211, footnote 1, and [24], [26], [30], [37], [41].

Forcing was done after my results, *e.g.* after [23], see [42], and my ideas was partly known to the creator of forcing in my lecture of 1958/9 years.[1]

,,The forcing development seems to be simple in the syntactical sense but I did not develop the syntactical theory in view of my very bad existence in Israel (see page 82)".

Those results began since 1954 year—based on my past experience in generalizations[2] of *Herbrand-Gödel's* proof of completeness theorem in two directions:

1. Finitization of all considerations by means of different conditions, [23], [26], [27], [40], [41].

2. Asympotic approximations[3], [22], [25], [28], [32]—[37], [41].

The third step of those results was open and dealing here with continuous values we begin with Boolean algebra of ones but a contrary to my predecessors I shall not restrict myself to propositional calculus and I shall give generalized models of the first order functional calculus.

Of course, the indication of continuous values was in *J. Słupecki's* lectures of 1950-7, *e.g.* about *J. Łukasiewicz's* infinite logics.

Thus we regard the closed interval [0, 1] as values of arbitrary formulas—simply: probabilistic values of formulas—and an infinite Boolean algebra, *i.e.* infinite Boolean logic, for the first order functional calculus with asymptoticly finite interpretation of quantifier Π leading to adequateness theorems with a very simple exterior form of generalizations of Herbrand's theorems and statistical decidability of the first-order functional calculus, [6]-[8], [13], [37], [41].

We recall the draft of the computational language used in my published papers—simply: computers—called first-order functional calculus:

(0. 1) Variables: (1′) free: $x_1, x_2, \cdots$(simply x);

(2′) apparent: $a_1, a_2, \cdots$(simply a);

(0. 2) Relation signs: $f^1_1, \cdots, f^1_q, f^t_1, \cdots, f^t_q$ with $t=q$;

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[1] I mean one of creators of forcing and my lecture of Jerusalem.

[2] New remarks, for my simple proof of completeness theorem was done independently on the above, see [22], with scientific and pedagogical domination including not published decidability theorems of certain formulas.

[3] I deal here with approximations of models what is impossible till my results; it is possible to find words of approximations in last publications of other authors dealing only with Herbrand's theorems but ones are in other sense, *e.g.* reduction to propositional calculus. All my results have a constructive and finite character.

(0. 3) Logical constants: $'$ (negation), $+$ (alternative), Π (general quantifier). The quantifier Σ is defined[1];

(0. 4) $h(E)$, $c(E)$—the number of free and apparent variables occurring in E, respectively;
$\{i_{h(E)}\}$—indices of all free variables occurring in E;
$\{i_m\}=i_1,\cdots, i_m$;

(0. 5) $n(E)=\max\{\{i_{h(E)}\}, h(E)+c(E)\}$, $n(U)=\max\{n(E)\colon E\in U\}$;

(0. 6) $E(u/x)$—the expression resulting from E by substitution of u for each x in E with known restrictions;

(0. 7) $C\{E\}$—the set of all parts of E;

(0. 8) M, $M_1,\cdots$—models; Q, $Q_1,\cdots$—non-empty set of models of the same power (for finite models it is also used the word: rank). $Q(k)$—Q is the set of models of power k—called: generalized model of the power k or simply: generalized model;

(0. 9) The pair $\langle D, \{F_t^q\}\rangle$ denotes a model, *i.e.* the domain D is an arbitrary non-empty set and $\{F_q^t\}$ is an arbitrary finite sequence of relations such that F_i^j is j-ary relation on D, $i=1,\cdots, q$ and $j=1,\cdots, t$.
A model of the power k is such model whose domain has exactly numbers $1,\cdots, k$ (k may be infinite), and then we write $V(M)=k$;

(0.10) For each model $M=\langle D, \{F_q^t\}\rangle$ by $M|s_1,\cdots, s_k|$—or simply:
$M|\{s_k\}$—we denote a model $\langle D_k, \{\Phi_q^t\}\rangle$ of the power k such that for each $r_1,\cdots, r_i\leqq k$ and $i=1,\cdots, t$ and $j=1,\cdots, q$:

$$\Phi_j^i(r_1,\cdots, r_i) \text{ iff } F_j^i(s_{r_1},\cdots, s_{r_i})\,.$$

So $M|\{s_k\}=\langle D_k, \{\Phi_j^i\}\rangle$; if $\{s_k\}$ is empty, then one holds for all models; $M|\{s_k\}$ is a submodel of M in the meaning of homomorphism;

(0.11) Meta-quantfiers: (K), $(\{K_m\})$, $(\exists K)$, $(\exists\{K_m\})$—for each K, $\{K_m\}$ and there exists K, $\{K_m\}$, respectively;

(0.12) $\{p_m\}=p_1,\cdots, p_m$—finite sequence of numbers in [0, 1];

(0.13) $'$, $\dot{+}$ —complemention and sum of Boolean algebras in the interval [0, 1]; we give two examples of last algebras, [14]-[16], [19], [33]-[37], [41], [48] (My proof rules, p. 82, give a simple assertion of ones.).
(1′) *Lukasiewicz's* one: $p'=2-p$, $p\dot{+}q=\min(1, p+q-1)$;

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[1] According to the above Π may be approximated by means of finite compositions of diodes and triodes—impossible till my results. An expression in which an apparent variable a belongs to the scope of two quantifiers Πa is not a formula; if a does not occur in E, then ΠaE is not a formula.

(2′) Algebra of probabilities: $p'=1-p$; $p\dot{+}q=1$ if p is the value of E and q is the value of E', $p\dot{+}q=p+q-pq$ otherwise and we interprete formulas E, E' as mutually exclusive events;

(0.14) w, $w_1, \cdots$—numbers 0, 1,; w^p_j—for instance: the j-th member of the binary expansion of the number p;

(0.15) $S(\{i_\tau\})$—the set of all atomic formulas with indices of free variables belonging to $\{i_\tau\}$; V_0—function on $S(\{k\})$ with values in [0, 1]; $\vee(\{p_m\})=k$ means: V_0 is only defined on $S(\{k\})$;

(0.16) m—the length of the considered binary expansions of numbers (it is here partly an editorial assumption).

The construction of the proper prime ideal [22], [28]—[32], gives the natural origin of my proof rules with cut-rule and its simplifications;

Let us recall ones:

(1.1) $$E+E' .$$

(1.2) $$\text{If } E+F, \text{ then } F+E .$$

(1.3) $$\text{If } (E+F)+G, \text{ then } E+(F+G) ,$$

(1.4) $$\text{If } E, \text{ then } E+F .$$

(1.5) $$\text{If } E+G, \text{ and } F+G', \text{ then } E+F$$ [2]

(1.6) $$\text{If } E+F,\ x \in C\{E\}, \text{ then } E+\Pi aF(a/x) ,$$

(1.7) $$\text{If } E+\Pi aF, \text{ then } E+F(x/a) .$$

In [32] I generalized the last construction with the proof rules according to generalized models, *i.e.* truncated inferences of families of sets of formulas; another kind of generalizations may be obtained in directions of [18], [21]—[36], [43]—[44].

So in view of the development of "forcing" I shall present it in the above forms describing kinds of [3], [5], [42] and let the reader identify here models with their descriptions:

For an arbitrary generalized model $Q(k)$, for an arbitrary model $M=\langle D_k, \{\Phi^t_q\}\rangle \in Q$, for an arbitrary formula E and each $\{i_\tau\}\supset\{i_{h(E)}\}$, $\tau+c(E)\leqslant k$, we present

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1) It is a Boolean algebra in [0, 2] and the reader will easy give other examples of Boolean algebras in [0, 1].

2) Regarding my cited paper it may be simplified to:
(1.5′) If $E+G$ and $E+G'$, then E.
(1.1)—(1.5) are rules of propositional calculus.

an inductive definition of the notion "M forces E"—in symbols: $M \Vdash E$—and it is given in a brief sketch of [5]:

(d1) $M \Vdash f_i^j(x_{i_1}, \cdots, x_{i_j})$ iff $F_i^j(i_1, \cdots, i_j) \in M$,

(d2) $M \Vdash F+G$ iff $M \Vdash F$ or $M \Vdash G$,

(d3) $M \Vdash F'$ iff $\sim(\exists M_1)\{(M_1 \in Q) \wedge (M_1 | \{i_\tau\} = M | \{i_\tau\}) \wedge (M \Vdash F)\}$,[1)]

(d4) $M \Vdash \sum aF$ iff $M \Vdash F(x/a)$.

If Q is closed under permutations, then x may be the first $x \in C\{F\}$, see [23], [24], [26], [30], [34], [38].

The basic lemma of truncation (the regarded homomorphismes) is:

L.1. $M | \{s_k\} | \{j_m\} = M | \{s_{k_{j_m}}\}$, see [9], [11]

D.1. $M_1 \in M[k]$ iff $(\exists\{s_k\})(M_1 = M | \{s_k\})$

$M[k]$ is the set of all models $M | \{s_k\}$ of the power k.

To formulate the equivalence of my invariance relations and forcing let us introduce my generalization of satisfiability definition according to [29], [31], [32], *i.e.* first we introduce inductively the following functional W with the above restrictions:

(1d) $$W\{k, Q, M, \{i_\tau\}, f_i^j(x_{i_1}, \cdots, x_{i_j})\} = 1 \text{ iff } F_i^j(i_1, \cdots, i_j) ,$$

(2d) $$W\{k, Q, M, \{i_\tau\}, F'\} = 1 \text{ iff} \sim W\{k, Q, M, \{i_\tau\}, F\} = 1$$
$$\text{iff } W\{k, Q, M, \{i_\tau\}, F\} = 0 ,$$

(3d) $$W\{k, Q, M, \{i_\tau\}, F+G\} = 1 \text{ iff } W\{k, Q, M, \{i_\tau\}, F\} = 1 \vee$$
$$\vee W\{k, Q, M, \{i_\tau\}, G\} = 1 ,$$

(4d) $$W\{k, Q, M, \{i_\tau\}, \Pi aF\} = 1 \text{ iff } (j)(M_1)\{(j \leqq k) \wedge$$
$$\wedge (M_1/\{i_\tau\} = M/\{i_\tau\}) \rightarrow W\{k, Q, M, \{i_\tau\}, j, F(x_j/a)\} = 1\}^{2)} ,$$

D.2. $N(k, Q, G)$ iff $(\{i_\tau\})\{(\{i_\tau\} \supset \{i_{h(G)}\}) \wedge (\tau + c(G) < k) \rightarrow (i)(M)(W\{k, Q, M, \{i_\tau\}, G\} = 1$

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1) Of course, speaking only about descriptions of models, domains of regarded models may be different ones, the segnence $\{i_\tau\}$ may be replaced by a number $r \geqq \max \{i_\tau\}$ and we can restrict ourselves to extensions of M instead of $M | \{i_\tau\} = M | \{i_\tau\}$, *i.e.* to suitable extensions of M cutted to r.

The rule "$M \Vdash x$ iff $x \epsilon D_k$" is here omitted.

2) The above definition of a switching function is simplified and the general form of (4d) is the following:

(4*d*) $W\{k, Q, M, \{i_\tau\}, \{z_\tau\}, \Pi aF\} = 1$ iff

$$(j)(x)(M_1)\{(x_j \text{ does not occur in } F) \wedge (j = 1, 2, \cdots) \wedge (z \epsilon D_k) \wedge (M_1/\{z_{i\tau}\} = M/\{z_{i\tau}\}) \rightarrow$$
$$W\{k, Q, M, \{i_\tau\}, j, \{z_\tau\}(z/j), F(x_j/a)\} = 1\} ,$$

where x_j is the name of z_j; the formulation (1*d*)-(3*d*) in the last case is immediately.

iff $W\{k, Q, M, \{i_\tau\}, i, H\}=1)\}$

As a basis of my probalistic models we give[1]

D.3. $F\in P(k, Q, M, \{i_\tau\})$ iff $(\exists H)\{(H\in C(F))\wedge(N(k, Q, H)\rightarrow W\{k, Q, M, \{i_\tau\}, F\}=1)\}$

D.4. $F\in P\{k, Q, M\}$ iff $F\in P(k, Q, M, \{i_{k(F)}\})$

D.5. $F\in P\{k\}$ iff $(Q)(M)\{Q(k)\wedge(M\in Q)\rightarrow(F\in P\{k, Q, M\})\}$,

D.6. $E\in P$ iff $(\exists k)\{(k\geqq n(E))\wedge(E\in P\{k\})\}$[2]

I cite from my papers the explanation of the last definitions:

$W\{k, Q, M, \{i_\tau\}, E\}=1$ may be read: the model M satisfies E respectively to Q and $\{i_\tau\}$.

If Q is one-elementing, then W is the usual satisfiability function in the domain of ordinary numbers of D_k; then D.2..4. create, obviously, the usual truth definition in M.

If M is a model and $Q=M[k]$, then elements of Q are submodels of M in the sense of homomorphism, the number j in (4d) is the name of an arbitrary element of the domain of M and *D.4.* says $\{i_\tau\}$ has not influence in whole on the introduced generalization of the satisfiability definition as in one-elementing Q, *i.e.* as in the case of a usual model; the invariant relation $N(k, Q, G)$ holds for connectives of propositional calculi—so in the case of practical verifications we need verify the invariant relation only for formulas of the form ΠaH, for some H; this relation asserts the same as in usual models, namely: the introduced generalization of the satisfiability definition depends only on values of free variables of E and it does not depend on the conditional sequence $\{i_\tau\}$ which does not determine here values of free variables—hence the name of Q as a generalized model.

D.5.-6. are pictures of the usual truth definition in its homomorphic generalization introduced above. For normal forms it suffices to guess only $H=E$ and the implication to the left instead of the second equivalence in *D.2.*; the last implication suffices for theses.

The main theorem:—P is the class of all theses of the first order functional calculus, *i.e.* of all true formulas of the calculus.

Its syntactical proof gives simultaneously completeness of infinite many

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1) They are not needed in the formulation of the equivalence lemma called: invariance lemma.

2) Instead of $C(E)$ the reader may regard a suitable set of formulas in *D.*3. respectively to his tasks.

The number $n(E)$ may be less than used here.

Boolean important calculi with quantifiers of finite interpretations which approximate the first order functional calculus and so the proof gives a simultaneous one of generalizations of Gödel, Skolem, Lövenheim, Herbrand's theorems which are here constructive ones for using generalized models we obtain only constructive proofs, but in more general sense regarded above.

Invariance lemma: —Let E be an arbitrary formula,

$$\{i_\tau\}\supset\{i_{h(E)}\},\ \tau+c(E)\leqq k \text{ and } Q(k);$$

then:

M forces E iff $W\{k, Q, M, \{i_\tau\}, E\}=1$ and $N(k, Q, G)$ for all $G\in C\{E\}$.

The simple proof of the last lemma is inductive respectively to the length of E and I received this lemma about 17 years ago proving the invariant relation N.[1]

In practical veryfications for E and given M we may restrict Q to extensions of $M|\{i_{h(E)}\}$.

Let us cite [38] the second time:

Let the reader formulate other ways, likely to force theory, leading to analogical invariance lemmas (for instance, according to my papers) and first of all let him formulate my weaker assumptions about quantifiers for normal formulas in notions of forcing, see cited papers.

The generalization for arbitrary languages —computers— is in print since several years.

Of course, and it is published in my papers, that my generalized models give generalizations of matrix methods, *i.e.* they only give constructive proofs of independence and consistency.

So about ten years ago —being also in contacts with *Jerzy Stupecki* —I wrote to *Karimiers Ajdukiewier* and *Karimiers Kuratowski*:

My results open a new start of reconstruction of the whole Mathematics on intuitive and constructive ways.

It is easy to see the connection of my results with group theory (of automorphismes), see [17], and it is known and it is seen in this paper that all results are also regarded in Game theory; so speaking about normal games with zero-sum it is important to regard complete matrices with very important results of *Jerzy Stupecki*, and *Bolestaw Sobocinski*, *e.g.* [43], [45], [48], and regarding *J.*

[1] It is immediately seen from the kind of my publication but, of course, I did not use the notion "forcing" and I only used the semantic, *i.e.* it is a lemma of an assumption for negation instead of Π; see the definition of $\sum$.

Stupecki's seminars in 1947/8, one of his pupils *M. Reichaw*, today also Professor of University and Polytechnic, solved his problem of complete matrices replacing *S. Nicod's* matrices by one $n\times n$ matrix. His inductive proof is very interesting, very clear and important and I plan to publish it in next journals of Creation in Mathematics [21].

The above gives the common origin of possible new ways —the common origin written in my papers, *i.e.* the historical ages problem of decidability formulated in my solutions:

Mathematical therems[1)] are semidecidable (*K. Gödel. S. Herbrand*), [1], [10], [12], [14], [15], [18], [22]—[35], [39], [41] undecidable (A. Church) [1], [10], [12], [16], [30] and statistically decidable [35]—[37], [39], [41].

Of course, my generalized models give also finite generalizations of the following:

If instead of formulas we regard equations, then using Poretzki's expansions we obtain linear polynomials and so the consistency is equivalent to existence of simultaneous solution of the corresponding polynomials. So, for instance, we obtain here generalizations of linear programming, see page 95.

Let us for brevity of the lecture—suppose an editorial assumption:

We only regard formulas of a given length and it may be done here in practical applications.

Introducing my generalized probabilistic satisfiability definition we explain several additional remarks about 0-1 sequences.

For a given n and $p\in[0, 1]$ let $(w_1^p,\cdots, w_n^p)$ denote a sequence of 0-1 numbers and let g be a function on the interval $[0, 1]$ with values $\{w_n^p\}$, constructed from p, and for instance, for each $p\in[0, 1]$:

1. $g(p)=\{w_n^p\}$ is the binary part-expansion of p of length n, where w_i^p is the i-th element of the binary expansion of p. Instead of the binary expansion we can take the decimal expansion or other one.

2. $\{w_n^p\}$ is the *Bernoulli's* sequence of n independent trials, *e.g.* if X is a binomial random variable, $n=6$ and $(w_1^p,\cdots. w_6^p)=(0, 1, 0, 0, 1, 1)$, then it may be:

$p=P\{X=2\}+P\{5\leqq X\leqq 6\}$, where $P\{i\leqq X\leqq j\}$ is the probability that $i\leqq X\leqq j$, see [37].

3. $\{w_n^p\}$ is generated by a matrice $\|a_{ij}\|$ with $|a_{ij}|\neq 0$ or another operator.

Let us point out, if we regard (in my papers) sets of models closed respec-

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1) It contains the termination problem in computers.

tively to permutations we may consider the number p as generator of numbers p^* with truncated permuted expansions; so each sequence $\{p_m\}$ of numbers determines certain generated prolongations $\{p_{m+1}\}$ putting, *e.g.* $p_{m+1}=p^*$, see [23], [26], [30]. The part-expansion $g(p_i)$ of p_i we write in i-th column of a matrice, $i=1,\cdots, n$, and so $\{p_m\}$ determines a matrice $n\times m$ of part-expansions of $\{p_m\}$ but for brevity we put $n=m$.

D.7. $(j\sim s,\ \{p_m\},\ \{i_\tau\}\leqq k)$ iff $(R)\{(R\in S(\{i_\tau\}))\wedge V_0\{k,\ R\}=p\in\{p_m\}\rightarrow(w_j^p=w_s^p)\}$.

And it asserts that j-th and s-th elements of suitable binary expansions are equal.

We extend the introduced V_0 to a composition of threshold function V defined for an arbitrary formula E with indices of free variables $\leqq k$ and $\{i_\tau\}\supset\{i_{h(E)}\}$:

(1D) $V\{k,\ \{p_m\},\ \{\tau_\tau\},\ R\}=V_0\{k,\ R\}\in\{p_m\},$[1]

(2D) $V\{k,\ \{p_m\},\ \{i_\tau\},\ F'\}=V\{k,\ \{p_m\},\ \{i_\tau\},\ F\},$

(3D) $V\{k,\ \{p_m\},\ \{i_\tau\},\ F+G\}=V\{k,\ \{p_m\},\ \{i_\tau\},\ F\}\dot{+}V\{k,\ \{p_m\},\ \{i_\tau\},\ G\}=1,$

(4D) $V\{k,\ \{p_m\},\ \{i_\tau\},\ \Pi aF\}=p$ iff $(j)\{(j\leqq m)\rightarrow(w_j^p=1$ iff $(s)(r)\{(s\leqq m)\wedge(r\leqq k)$ $\wedge(j\sim s,\ \{p_m\},\ \{i_\tau\}\leqq k)\wedge V\{k,\ \{p_m\},\ \{i_\tau\},\ r,\ F(x_r/a)\}=p_r\rightarrow(w_j^{p_r}=w_s^{p_r}=1)\})\}.$

(4D) is an adequate form of (4d) by means of numbers of the interval [0,1] and $n=m$ is the number of elemenls of Q.

$V\{k,\ \{p_m\},\ \{i_\tau\},\ E\}=p$ is read: the generalized probabilistic model $\{p_m\}$ gives the value p for E respectively to $\{i_\tau\}$ and k; if numbers $\{p_m\}$ are only 0, 1, then V is a form of usual satisfiability function in domains of numbers $\leqq k$.

In applications we may restrict ourselves to finite Boolean algebras and my all definitions have an algorithmic character with statistical computation, [37], [39], [52]:

D.8. $J(k,\ \{p_m\},\ G)$ iff $(\{i_\tau\})\{(\tau+c(G)<k)\wedge(\{i_{h(G)}\}\subset\{i_\tau\})\rightarrow(j)(V\{k,\ \{p_m\},\ \{i_\tau\},\ G\}$ $=V\{k,\ \{p_m\},\ \{i_\tau\},\ j,\ G\})\}.$

The last invariance relation J reduces the influence of $\{i_\tau\}$ to $\{i_{h(G)}\}\subset\{i_\tau\}$, see page 84.

D.9. $E\in P\{k,\ \{p_m\}\}$ iff $(\exists G)\{(G\in C\{E\})\wedge(J(k,\ \{p_m\},\ G)\rightarrow V\{k,\ \{p_m\},\ \{i_{h(E)}\},\ E\}=1)\}$

My results give immediately, [32], [37]:

Adequacy theorem: If U is an arbitrary set of formulas, the U is consistent iff there exists such $\{p_m\}$ that if $E\in U$, $k\geqslant n(E)$ and $(\{p_m\})=k$, then $E\in p\{k,\ \{p_m\}\}$ and $E'\in p\{k,\ \{p_m\}\}$.

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[1] So m must be sufficient great and $m\leqq 2^{qtk^t}$

Additionally: if $U \vDash F$ and $k \geqq \max\{n(F), n(U)\}$,[1] then $F \in P\{k, \{p_m\}\}$.

According to the last theorem each consistent set u of formulas is defined by a pair $\langle n(U), \{p_m\}\rangle$, $n(U) \leqq k$, or omitting the infinity it is defined by a sequence $\langle\{p_m\}, \{p_{m+1}\}, \{p_{m+2}\}, \cdots\rangle$ with finite m:

The last sequence $\{p_m\}$ is called "adequacy U-sequence."

The adequacy theorem may be written shortly in symbols of a form of generalization:

A-theorem: $-U$ is consistent. iff:

$$(\exists\{p_m\})\{(V(\{p_m\})=k)\wedge(E)\{(U\vdash E)\wedge(k\geqq n(E))\rightarrow(E\in P\{k, \{p_m\})\wedge(E'\in P\{k, \{p_m\}\})\}\wedge$$
$$\wedge\{(U\vDash F)(k\geqq \max\{n(F), n(U)\})\rightarrow(F\in P\{k, \{p_m\}\})\}\}$$

A-theorem for U empty contains the main theorem of preceding lecture, *e.g.* in the following statistical formulation introducing statistical tests:

Let us suppose E is the measured quantity and for brevity let 1 be its theoretical value. Our tests indicate great numbers k and let P_j be probabilistics (our confidences) with $\lim_{j\to\infty} P_j=1$ and let $\lim_{j\to\infty} e_j=0$.

According to history of statistics let us take an intuitive basis assuming E has normal density; then we guess:

D.10. $E\in P$ iff $(j)(\exists k)\{(k\geqq n(E))\wedge(\{p_m\})\{(1\leqq m\leqq 2^{qtk^t})\rightarrow(\exists G)\{(G\in C\{E\})\wedge$
$(J(k, \{p_m\}, G)\rightarrow V\{k, \{p_m\}, \{i_{h(E)}\}, E\}\in[1-e_j, 1]$ with the confidence $P_j)\}\}\}$

It may be proven [37]:

Main theorem with statistical tests:—P is the class of true formulas of the first- order functional calculus, *i.e.* the class of theses of that calculus.

So D.10. gives immediately statistical tests in the last main theorem. And omitting the quantifier (j) with the replacement of the interval $[1-e_j, 1]$ by the number 1 in D.10. we obtain my published results in different journals but in notion of probabilistic models and without statistical tests, see pages 90-95.

Let us point out, regarding only usual models we can only restrict ourselves to $k\geqq\aleph_0$ and two values 0, 1 of V; but the last adequacy theorem may only deal with finite k what is impossible for usual models.

The main theorems may be also completed:

The quantifier $(\{p_m\})$ in D.10. is restricted to all adequacy sequences with generated prolongations, *e.g.* for $m\geqq 2^{qtk^t}$ with $k\geqq n(U)$.

In the sequel we only regard adequacy U-sequences and so the introduced

[1] If $n(U)$ is infinite, then m, k are infinite; but $n(U)$=infinity may be omitted in that theorem writing: k sufficient great; if we restrict ourselves to formulas of a given length then $n(U)$ is finite.

values $V\{k, \{p_m\}, \{i_{h(E)}\}, E\}$ in D.9., D.10 and in last theorems may be done independent on k by means of the definition:

(1) $P^U\{E\} = \lim\limits_{k\to\infty} V\{k, \{p_m\}, \{i_{h(E)}\}, E\}$

(If $\{p_m\}$ depends on k, then we also assume (1).)

We say E, F are disjoint respectively to U iff $(EF)'$ is a thesis respectively to U.

A topological space of formulas may be here obtained in usual ways (*e.g.* $P^U\{E\}-p^U\{F\}$) but we do not use properties of a metric and therefore we shall speak about metric in the sense of the following definition:

D.11. $D(E, F) = P^U\{EF'+E'F\}$[1]

Hence for instance denoting $P\{E\}=p$, $P\{F\}=q$ we have in the probabilistic model, page 81,

$$D(E, F) = \begin{cases} 0, \text{ if } E\equiv F \\ p+q-2pq, \text{ if } E\not\equiv F^{2)} \end{cases}$$

The explained infinite probabilistic models (but not finite one) where regarded *e.g.* by A.N. Kolmogoroff, J. Lós, A. Mazurkiewicz, R. Suszko,$\cdots$)

The limit of a sequence E_i of formulas is defined usually:

D.12. $E = \lim\limits_{i\to\infty} E_i$ iff $\lim\limits_{i\to\infty} D(E_i, E)=0$

From the above follows respectively:

Convergence theorem:— $\lim\limits_{i\to\infty} P\{E\equiv E_i\}=1$ iff $\lim\limits_{i\to\infty} E_i=E$ iff $(\exists N)(i)\{(i\geq N) \to (U\vdash E_i\equiv E)\}$.

Generalization of Herbrand's theorem:— $\lim\limits_{i\to\infty} E_i=E$ iff there exists n such that $U\vdash E_n\equiv E$.[3]

So we obtain immediately:

Convergences of probabilities:— $\lim\limits_{i\to\infty} E_i=E$ iff $\lim\limits_{i\to\infty} P\{E_i\}=P\{E\}$.

And we obtained a usual convergence and not only the strong law of great numbers.

It is interesting that the last formulation of theorems contains Herbrand's theorems[4] for an arbitrary consistent sets of formulas according to the following

---

1) It is the symmetrical difference: $E\dot{-}F=(E\equiv F)'$ and it is an example of non threshold function.

2) Of course, the introduced asymptotic probability satisfies A. Mazurkiewicz's axioms of probabilities but the proof is based on proved theorems in my published papers.

3) We do not regard here syntactical proof rules, see pages 81-82.

4) Those formulations belong to the author and we may obtain here also Craig, Gentzen and A. Robinson's theorems.

important examples of convergent sequences of formulas:

1. Herbrand's quantifierless formulas for an arbitrary expression creating finite alternatives E_m of infinite many formulas; it suffices to see this theorem for theses and then according to the above the generalization is also seen.

2. My formulas without quantifiers determined by main theorems and D.6.

3. My generalization of Herbrand's formulas in a simultaneous proof of Gödel, Skolem, Lövenheim and Herbrand's theorems with a new notion: E is a thesis of a set of formulas A respectively to B, $A \in B$; the syntactic way is published in [32] and it is a generalization of [22].

We emphasize:

A-theorem gives probabilistic tables for Boolean calculi with quantifiers of finite interpretations which approximate the first order functional calculus (with added axioms U else; additionally it gives an infinite probabilistic table).

In the following theorem we identify formulas with events and the equivalence with the identity and we deal with the probabilistic model; so from the infinite addition axiom of probabilities follows immediately:

Kolmogoroff's theorem (generalization of the infinite axiom of probabilistics): —If U is consistent and events E_i, E_j are disjoint respectively to U, $i \neq j, i, j = 1, 2, \cdots$, and $\lim_{m \to \infty} (E_1 + \cdots + E_m) = E$, then:

$$P\{E\} = P\{E_1\} + \cdots + P\{E_m\} + \cdots$$

Though it is a reformulation of Kolmogoroff's theorem it is a new theorem in view of my generalized models.

Values of formulas and statistical testing:—We shall describe different test methods and so first of all the measured quantity is an arbitrary formula E.

My published paper, see cited ones, describe different formulas E_i of propositional calculus which asymptotically replace E and even more, see pages 93-95.

First we regard the aforementioned different convergent sequences E_i with $\lim_{i \to \infty} E_i = E$.

And we have here three questions:

1. Estimation of E.
2. Comparison of different estimations.
3. Decisions.

Estimations of the value of E may be obtained on the following three ways: normal formulas (or especially Gödel-Herbrand's quantifierless formulas) with their propositional approximation, ones in the general case, my different approximations with finite-natural change of the interpretation of quantifiers and not of the structure of formulas in additional descriptions.

Comparison of two calculated estimations we obtain, *e.g.*, using statistical tests or Student's density and so we assume the measured quantity has the normal density.

We may here consider all questions of statistics with *e.g.*, Borel-Cantelli's lemma (giving interesting decidability criterions of arbitrary formulas), sequential analysis and recurrentive events[1] with all densities but we shall describe here stronger results and first of all we present in decisions a simple example of statistical tests [6]-[8], [50], [51].

Let us return to my above propositional approximations E_k of E or (Herbrand's ones wholly of another kind) and let us guess for brevity: E is a true formula; then according to the above we can have for instance: M formulas E_i are true—a finite machine jotted down N formulas in a program without a reply—and so $N—M$ formulas E_i are false $i=1,2,\cdots, N$.

Interpreting true E_k as white cards and false E_k as black ones and closing all cards in a box (we suppose a black box) we apply the known statistical choice with the probability of chosing k white cards between n chosen cards:

$$P_{NM}^{nk}=\frac{\binom{M}{k}\cdot\binom{N-M}{n-k}}{\binom{N}{n}}$$

For great N it is replaced by the binomial density with $p=M/N$ and afterwards by the normal density with the formula

$$P\left\{\left|\frac{k}{n}-p\right|<t\right\}=2\Phi\left(\frac{t}{\sigma}\right)-1=P\,,$$

where P is the confidence and $\bar{\sigma}=\sqrt{p(1-p)/n}$.

The most interesting case p is very small; then applying Poisson's density

$$P_\lambda(k)=\frac{\lambda^k}{k!}e^{-\lambda},\ \lambda\approx np\ .$$

with $\lambda\to\infty$ we use the normal density with $\bar{\sigma}=\sqrt{\lambda/n}$ and then we obtain the following computation results:

---

1) We recall a formula in directions of the binomial density:

$$P\{X_n\leqq k\}=P\{S_k\geqq N\}\,,$$

where X_n takes on values of strict outcomes of the recurrentive event x in n regarded trials and S_k takes on values of numbers of trials till the k-th outcome of x.

If $P=0.95$, $\lambda=0.0001$, $t=0.00001$ then $n=384.16\cdot 10^{12}\approx 22.3\cdot 10^{5}$ distances between the earth and the sun.[1)]

In sequential analysis using formulas of [51] (without errors of approximations) we obtain for the errors of the first and second kinds $\alpha=0.01$, $\beta=0.05$, respectively, with the hypothesis H_0: $\lambda_0=0.0001$.

1. against the alternative one H_1: $\lambda_1=0.00021$ for the current most powerfull test:

$$n=\left(\frac{t_{0.5}-t_{0.1}}{\lambda_1-\lambda_0}\right)^2=\left(\frac{2.32-1.65}{10^{-5}(21-10)}\right)^2=\frac{0.44}{121}\cdot 10^{10}=371\cdot 10^{5}\,,$$

2. against the alternative H_1: $\lambda_1=10^{-4}$ calculated by means of approximation in the interval $[9\cdot 10^{-5},\ 2\cdot 10^{-4}]$ with the most powerfull probability ratio test: $n=382\cdot 10^{4}$. Let the reader read distributions of calculated n and a better calculation of errors of the first and second kinds.

Another estimation of n may be obtained restricting ourselves only to the binomial density with results of [49] instead of the above Poisson's approximation.

The sequential Probability Ratio Test gives methods of construction of equivalent formulas of the first-order functional and[2)] a resolvent of a given formula.

Last results are called: weak statistical decidability of theorems.

The great importance of all results is in computation by means of computers and we cite [2] of mathematical conference of last years:

"*2. Counter examples*:—Computing machines have been of service to mathematics in printing out, by example, the falsity of a conjectured result. The theory of Numbers is a good branch of mathematics to look for these counter examples not only because of its discrete variable nature but also because the subject abounds with easily stated propositions whose truth values are very difficult to establish.

A very recent discovery of Lander and Parken and the CDC 6600 is a case in point. Euler (circa 1769), in discussing the Fermat Problem, declared that is also impossible to find three fourth powers whose sum is a fourth power or to find four fifth powers whose sum is a fifth power. Two centuries later, this

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1) The convergence of Poisson's density to the normal one is in [37] as a conclusion of my general theorem with a simple proof and it states that the normed Poisson's random variable has asymptotically the normal density with the expectation 0 and variances 1 and even more for the local theorem.

2) Ones are not simpler.

summer, the 6600 came up with the undisputable fact that

$$27^5+84^5+110^5+133^5=144^5$$

This not only demolishes Euler's assertion about fifth powers but even raises hopes that the equation

$$x^4+y^4+z^4=w^4$$

may also be solvable. I know three different computing establishments where this equation is now under scrutiny.

Not all searches for counter-examples have triumphial endings. In fact, in the case of some famous unsolved problems, it would seem foolhardy to invest much good machine time in such a research. Examples are the Four Color Problem and the Goldbach Conjecture that every even number (>2) is the sum of two primes.

3. Verification of conjectures: When one has been unsuccesful in finding a counter-example, one can sometimes report that one's machine time was well spent in verifying the truth of N cases of the general proposition one secretly was hoping to demolish. If N is pretty large, one can say that the proposition is now more plausible than before. A metric for plausibility has never been proposed, as far as I know, but if we had one we may be sure that it would not be a linear function of N''.

Thus results of the paper complete the last known facts and it is anticipated that an understanding of presented algorithmes with weak and strong statistical decidability of theorems will permit many scientists and readers to formulate and to solve problems which they currently believe unsolvable.

Describing further strong decision methods we shall sketch parts of my lecture in Computers Conference of Menaggio (2.8—14.8. 1970), Italy), Paris (22.9. 1970, France) and Rome (5—20.10. 1970, Italy).[1)]

First of all we recall my first published truncated satisfiability (composition of treshold) function of [22], [34] with suitable truth definitions:

For an arbitrary $M \in Q(k)$ and every formula E with indices of free variables $\leqq k$ we have my first inductive definition of the truncated satisfiability:

(id 1) $W\{M, Q, f_j^m(x_{r_1} \cdots, x_{r_m})\}=1$ iff $F_j^m(r_1, \cdots, r_m)$,

(id 2) $W\{M, Q, F'\}=1$ iff $\sim W\{M, Q, F\}=1$ iff $W\{M, Q, F\}=0$

(id 3) $W\{M, Q, F+G\}=1$ iff $W\{M, Q, F\}=1 \vee W\{M, Q, G\}=1$

1) Weak ones was also presented in my visiting lectures of Milano, Pavia and Venezia, Italy, 1970 year.

(id 4) $W\{M,\ Q,\ \Pi aF\}=1$ iff $(i)(M_1)\{(i\leqq k)\wedge(M_1\in Q)\wedge(M_1|\{i_{k(F)}\}=M/\{i_{k(F)}\})$
$\rightarrow W\{M,\ Q,\ F(x_i/a)\}=1\}$

D.13. $n(Q,\ k)$ iff $Q(k)\wedge(\{t_k\})(M)\{(\{t_k\}\leqq k)\wedge(M\in Q)\rightarrow(M/\{t_k\}\in Q)\}$[1)]

D.14. $N(Q,\ k)$ iff $m(Q,\ k)\wedge(t)(M_1)(M_2)(\exists M)\{(t+2\leqq k)\wedge(M_1,\ M_2\ \in Q)\wedge$
$\wedge(M_1|\{t\}=M_2/\{t\})\rightarrow(M\in Q)\wedge(M/\{t+1\}=M_1/\{t+1\})\wedge(M/\{t\},$
$t+2/=M_2/\{t\},\ t+2/)\}$

D.15. $E\in P\{Q,\ k\}$ iff $(M)\{(M\in Q)\rightarrow W\{M,\ Q,\ E\}=1\}$

D.16. $E\in P\{k\}$ iff $(Q)\{N(Q,\ k\}\rightarrow(E\in P(Q,\ k)\}$

D.17. $E\in P$ iff $E\in P\{n(E)\}$

If we regard elements of Q with an infinite sequence of monadic relations (added to those models for the relation $N(Q, k)$), then it is proved the main theorem, page 84, but then Q has infinite number of finite models of the rank $n(E)$.

Using probabilistic models we may omit the last infinity in the following manner:

Let us arrange all atomic formulas of $S(n(E))$(*i.e.* a finite number) and all elements of $Q(n(E))$ in a table with 0-1 values in the following table.

	R_1	$R_2\cdots R_s$
M_1	0	$1\cdots0$
M_2	0	$0\cdots1$
.	.	.....
M_i	1	$0\cdots1$
.	.	.....

The 0-1 sequence below R_i, $i=1,\cdots,s$ is regarded as binary expansion of a number of [0, 1] and so we have a finite number of numbers belonging to [0, 1]; in that construction we can realize first of all, the set $Q(n(E))$(without additional monadic relations) in the first $2^{qt(n(E))^t}$ expansion of those numbers$\cdots$

So the finite number of elements of [0, 1] generate the regarded set $Q(n(E))$ with additional monadic relations.

Thus according to my aforementioned visiting lessons the question is:

How to shoot the finite number of numbers belonging to [0, 1] and according to the reduction of the decidability problem it suffices to regard atomic formulas indicated by one apparent variable my theorems gives intervals tending to 0 containing ones and we have at least statistical approximation of those numbers.

1) Using arbitrary names D-13 may be omitted.

So closing the third question about decisions we can use here not only tests and Student's density in comparison of two results or *e.g.*, with the expected value 0, but also to use minimization of functions with their densities (and here I cite P. Erdös' and M. Kac's results [4] about normal limit density), linear expansions of formulas according to the generalization of Poretzki's formula in the interval [0, 2] with linear interpolation (if we only regard the interval [0, 1]) and extrapolation (if we regard the interval [0, 2]) of values, [48]: therefore we recall the expansion of formulas of propositional calculus:

$$f(p_1,\cdots, p_n)=\sum s_1\cdots s_n f(d_1,\cdots,d_n)$$

with $\sum s_1\cdots s_n=1$, $0\leqq s_i\leqq 1$, and

$$s_i=\begin{cases} \left.\begin{array}{r} p_i, \text{ if } d_i=1 \\ 1-p_i, \text{ if } d_i=0 \end{array}\right\} \text{ and } 0\leqq p_i\leqq 1 \\ \left.\begin{array}{r} p_i-1, \text{ if } d_i=2 \\ 2-p_i, \text{ if } d_i=1 \end{array}\right\} \text{ and } 1<p_i<2\,, \end{cases}$$

where the summation is over all the configurations of either $d_i=0$ or 1 or 2 and the coefficients $s_1\cdots s_n$ may be regarded as probabilities of occurrences of the corresponding independent random variables $d_1,\cdots,d_n$ and so we have here the Monte Carlo method with construction of random variables [37].

First experimental values of E are best given according to my generalized models of E, [23]-[27], [30]-[36], [39], and it is immediately seen that their construction is wholly new one.

If in a random verification it happens we rejected a true theorem E with a given confidence (but never vice versa, if it is the question), then we apply the following decision:

Reject another theorem with the same confidence, even if the formula was only rejected in inference conclusions on the basis of the rejection of E.

If the last way gives a contradiction in the above process, then we proved E; if no, then we reject both formulas with the same probability.

Hence the functional calculi—mathematical theorems—are strongly statistically decidable...

Regarding $\aleph_0$ propositional calculus we can construct syntactical proof rules to the last probabilistic semantic and to prove new completeness theorems based in my publications, see cited ones.

Thus speaking about switching functions, threshold ones, resolvents, Chow's parameters and linear programming we have here a natural generalizations of ones with the inclusion of the decidability problem in linear programming with applications of computers in our calculations and it will be a topic of my future papers, see [2], [4], [6], [13], [19], [20], [21], [23], [30], [36], [37], [39], [41], [43], [45], [46], [47], [48], [51], [52].

At last my problem from Poland—of about 17 years ago—was unsolved in view of my very bad conditions in Israel, [38]:

Finitization of functions:—Cut functions and replace my generalized models by means of generalized models with functions on finite sets with finite number of values.

Of course, the positive solution is published in my papers by means of reduction of functions to relations but the purpose of my future publications is to give derivative methods and so the positive solution with applications dealing with a direct method omitting reduction to relations, will be published in next my papers.

We close the paper citing my other important results in print:

First of all, according to my theorems the invariant relation can be omitted in the following way:

Let $i(G)=\max\{i_{w(G)}\}$, $m(E)=\max\{p(G)\leftarrow i(G)$, for each simple member G of $E\}$, where G is a simple member of E iff $E=F_1+G+F_2$, for some F_1, F_2 and G is not an alternative of formulas, $j(G)=\max\{k-p(G),\ i(G)\}$; then;

For an arbitrary generalized model $Q(k)$, for an arbitrary $T=\langle B_k,\ \{F_j^i\}\rangle\in Q$ and for an arbitrary formula E whose indices of free variables are $\leqq k$, we introduce the following inductive definition of the functional V_1:

(1d) $V_1\{Q,\ T,\ f_j^m(x_{r_1}\cdots,\ x_{r_m})\}=1$ iff $F_j^m(r_1,\cdots,\ r_m)$,

(2d) $V_1\{Q,\ T,\ F'\}=1$ iff $\sim V_1\{Q,\ T,\ \{i_t\},\ F\}=1$ iff $V_1\{Q,\ T,\ F\}=0$,

(3d) $V_1\{Q,\ T,\ F+G\}=1$ iff $V_1\{Q,\ T,\ F\}=1\vee V_1\{Q,\ T,\ G\}=1$,

(4d) $V\{Q,\ T,\ \Pi aF\}=1$ iff $(i)(T_1)\{(i\leqq j(F)+1)\wedge(T_1\in Q)\wedge$
$\wedge(T_1|\{j(F)\}=T|\{j(F)\})\rightarrow V_1\{Q,\ T_1,\ F(x_i/a)\}=1\}$

D.18. $E\in P_1\{k\}$ iff $(Q)(T)\{Q(k)\wedge(T\in Q)\rightarrow V_1\{Q,\ T,\ E\}=1\}$

D.19. $E\in P_1$ iff $(\exists k)\{(k\leqq m(E))\wedge(E\in P_1\{k\})\}$

The main theorem is; P_1 is the class of all theses.

Thus according my published results we obtain here simple generalized sequent rules with strong generalizations of the resolution method leading to better decidability methods (partially recursive functions).

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