A CHEBYSHEV SERIES METHOD FOR THE NUMERICAL SOLUTION OF FREDHOLM INTEGRAL EQUATIONS WITH ASSOCIATED EIGEN-VALUE PROBLEM.

By

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Abstract: An investigation has been made into the numerical solution of non-singular Fredholm integral equations by obtaining the expansion of the unknown function $f(x)$ in a series of derived Chebyshev polynomials. The solution of the eigen-value problem i.e. to find those values of the parameter for which the homogeneous integral equation possesses non-trivial solution has also been considered.

1. Introduction:

The use of a series of Chebyshev polynomials $T_n(x)$ to obtain the numerical solution of integral equations has been investigated by *Elliott* [1]. He considers the equation

$$f(x)-\lambda\int_{-1}^{1} k(x,y)f(y)dy=F(x)\ , \tag{1}$$

where $f(x)$ is the function to be determined. The constant λ, the kernel $k(x,y)$ and the function $F(x)$ are known. The eigen-value problem associated with the homogeneous integral equation has also been studied by Elliott in [2]. In this paper it has been shown that a series of derived Chebyshev polynomials $T'_n(x)$ can also be used effectively in such evaluations. The reason for preferring a series of $T'_n(x)$ in the approximation of $f(x)$ is that, since the maximum modulus of $T'_n(x)$ is n^2, the coefficients must decrease faster than $1/n^2$ to give convergence. The method has been verified by means of numerical examples. An upper bound of the error of such approximation to the solution of integral equation can however be deduced as in [3].

2. Method of solution:

To obtain the solution of (1), we suppose for convenience that the range of values of (x, y) is normalised so that $-1 \leq x,\ y \leq 1$ and that the kernel $k(x, y)$ is bounded. Following Elliott [1] we consider in brief the method of solving integral equations both for separable and non separable kernels.

(a) Separable kernel:

Let the kernel be separable so that

$$k(x, y)=\sum_{m=1}^{M} p_m(x) q_m(y) , \tag{2}$$

Then (1) becomes

$$f(x)=F(x)+\lambda \sum_{m=1}^{M} p_m(x) \int_{-1}^{1} q_m(y) f(y) dy , \tag{3}$$

Since $F(x)$, $p_m(x)$ and $q_m(y)$ are known functions we assume that of them the first two can be expanded in $T_n'(x)$ as in [4] and the last one in $T_n(y)$. These expansions can be found very easily.

We now look for the solution $f(x)$ in the form

$$f(x)=\sum_{n=1}^{N} a_n T_n'(x) \tag{4}$$

Now if

$$q_m(y)=\sum_{n=0}^{\infty}{}' c_n^m T_n(y) ,$$

where the prime denotes that the first term is halved, then the integral

$$I_m=\int_{-1}^{1} q_m(y) f(y) dy , \tag{5}$$

is a constant depending upon $a_1, a_2, a_3 \cdots a_N$ and can be evaluated by using

$$\left\{ \begin{aligned} \int_{-1}^{1} T_n'(y) T_s(y) dy &= 0 \qquad \text{if } n+s, \text{ be even} \\ &= \frac{2n^2}{n^2-s^2} \quad \text{if } n+s, \text{ be odd} \end{aligned} \right. \tag{6}$$

If b_n, d_n^m denote the coefficients of $T_n'(x)$ in the expansion of $F(x)$ and $p_m(x)$ respectively then substituting (5) in (3) and equating the coefficients of $T_n'(x)$ from both sides we get

$$a_n=b_n+\lambda \sum_{m=1}^{M} I_m d_n^m , \quad n=1(1)N , \tag{7}$$

where $$I_m=I_m(a_1, a_2 \cdots a_N) .$$

Equation (7) is a system of N linear equations for the N unknowns $a_1, a_2 \cdots a_N$. These can be solved numerically by standard methods.

(b) Non-separable kernel:

In this case we take $f(x)$ as in (4). The N constants $a_1, a_2, \cdots a_N$ are determined by writing down the integral equation at each of the N-points x_i, $i=1, 2 \cdots N$.

Equation (1) is then replaced by N equations

$$f(x_i)=F(x_i)+\lambda\int_{-1}^{1} k(x_i, y)f(y)dy \; . \tag{8}$$

Now for each x_i, the kernel $k(x_i, y)$ is approximated by a polynomial of degree M as in [1] in the form

$$k(x_i, y)=\sum_{n=0}^{M}{}' b_n(x_i)T_n(y) \tag{9}$$

With (4), (6) and (9) the integral

$$I(x_i)=\int_{-1}^{1} k(x_i, y)f(y)dy \; , \tag{10}$$

can be evaluated in terms of the coefficients. Since $F(x_i)$ is known, then using tables of $T_n'(x)$ [5] we can write down $f(x_i)$ in terms of $a_1, a_2 \cdots a_N$ for each value of x_i.

Equation (8) becomes

$$f(x_i)=F(x_i)+\lambda I(x_i) \; , \quad i=1(1)N \tag{11}$$

which is a system of N linear equations for the N unknown coefficients and can be solved very easily.

3. Numerical examples:

In this section we consider first an example of an integral equation in which the kernel is non-separable. The case for separable kernel can be handled quite easily. However in the second example we discuss an eigen-value problem associated with the homogeneous integral equation in which the kernel is separable.

i) $$f(x)\pm\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}f(y)dy=1 \; ,$$

Let us consider first the equation with positive sign and approximate the function $f(x)$ by a polynomial of sixth degree. Since $f(x)$ is an even function of x, we write

$$f(x)=a_1+a_3T_3'(x)+a_5T_5'(x)+a_7T_7'(x) ,$$

and consider only positive values of x_i at

$$x_i=0,\ 0.5,\ 0.8,\ 1.0.$$

Then using table 2 of [1] we get the coefficients $b_n(x_i)$ in the expansion of $k(x_i, y)$ as in (9) at the specified points x_i for $M=14$. Having found these coefficients for $k(x_i, y)$, $I(x_i)$ is evaluated for each value of x_i from (10) and then substituting those values in (11) we obtain the following system of equations

$$4.712390a_1-\ 8.986728a_3+16.806710a_5-\ 21.043288a_7=3.141592 ,$$

$$4.588030a_1+\ 0.909630a_3-14.682010a_5+\ 23.101484a_7=3.141592 ,$$

$$4.402688a_1+15.980146a_3-\ 0.889336a_5-\ 34.672301a_7=3.141592 ,$$

$$4.248740a_1+29.639646a_3+79.687550a_5+155.128218a_7=3.141592 ,$$

the solution of which gives

$$f(x)=0.682902+0.008404T_3'(x)-0.000080T_5'(x)-0.000018T_7'(x) ,$$

Taking the integral with negative sign and proceeding as before

$$f(x)=1.844448-0.024166T_3'(x)+0.000454T_5'(x)+0.000029T_7'(x) .$$

The comparison of these solutions with those obtained by *Fox* and *Goodwin* and *Elliott* is given in table 1 and 2. The results of *Fox* and *Goodwin* [6] have been presented only to $4D$ with an estimated maximum error of 1×10^{-4} due to round off and we see that the results obtained by this method agree exactly with those of *Fox* and *Goodwin* and with *Elliott* to within the prescribed error.

Table 1.

$$f(x)+\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}f(y)dy=1$$

x	Fox and Goodwin to $4D$	Chebyshev 6th degree Elliott.	Legendre 4th degree	Derived Chebyshev 6th degree Present method to $5D$
0	0.6574	0.65741	0.65745	0.65742
±0.25	0.6638	0.65385	0.66397	0.66384
±0.5	0.6832	0.68318	0.68323	0.68318
±0.75	0.7149	0.71482	0.71432	0.71486
±1.0	0.7557	0.75571	0.75576	0.75566

Table 2.

$$f(x)-\frac{1}{\pi}\int_{-1}^{1}\frac{1}{1+(x-y)^2}f(y)dy=1$$

x	Fox and Goodwin to 4D	Chebyshev 6th degree Elliott.	Legendre 4th degree	Derived Chebyshev 6th degree Present method to 5D
0	1.9191	1.91903	1.91925	1.91901
±0.25	1.8997	1.89958	1.89966	1.89957
±0.5	1.8424	1.84240	1.84261	1.84238
±0.75	1.7520	1.75208	1.75318	1.75198
±1.0	1.6397	1.63971	1.63987	1.63972

ii)
$$f(x)=\lambda\int_0^1 k(x, y)f(y)dy ,$$

where
$$k(x, y)=x(1-y) \quad \text{for} \quad y\geq x$$
$$=y(1-x) \quad \text{for} \quad y\leq x$$

This is an eigen-value problem in which we want to find those value of the parameter λ for which the homogeneous equation possesses a non-trivial solution. The analytical solution being $f(x)=\sin \pi x$ with $\lambda=\pi^2$. The kernel here is separable. Since the range is [0, 1]. We use a derived series of shifted Chebyshev polynomials for the series expansion of $f(x)$.

Now

(12) $$f(x)=\lambda\{(1-x)I(x)+xJ(x)\} ,$$

where

$$I(x)=\int_0^x yf(y)dy , \qquad J(x)=\int_x^1 (1-y)f(y)dy .$$

Let

$$f(x)=\sum_{n=1}^{\infty} A_n T_n^{*\prime}(x) \quad \text{and} \quad I(x)=\sum_{n=1}^{\infty} \alpha_n T_n^{*\prime}(x) .$$

Then using the following relations

(13) $$xT_n^{*\prime}(x)=\frac{1}{2}T_n^{*\prime}(x)+\frac{n}{4}\left[\frac{T_{n+1}^{*\prime}(x)}{n+1}+\frac{T_{n-1}^{*\prime}(x)}{n-1}\right], \quad n>1$$

(14) $$T_n^*(x)=\frac{1}{4}\left[\frac{T_{n+1}^{*\prime}(x)}{n+1}-\frac{T_{n-1}^{*\prime}(x)}{n-1}\right], \quad n>1$$

and $T_n^*(x)=\cos n\theta$ with $2x-1=\cos\theta$ it can be easily seen from the integral for $I(x)$ that

(15) $$\alpha_n=\frac{A_{n-1}-A_{n+1}}{8n}+\frac{(n-2)(n+1)A_{n-2}+2nA_n-(n+2)(n-1)A_{n+2}}{16n(n-1)(n+1)}\ , \quad n>1$$

and the condition $I(0)=0$ gives

(16) $$\alpha_1=\sum_{n=2}^{\infty}(-1)^n n^2\alpha_n\ .$$

Similarly if β_n denotes the coefficient of $T_n^{*\prime}(x)$ in the expansion for $J(x)$, then it can be easily seen

(17) $$\begin{cases} \alpha_n-\beta_n=\dfrac{A_{n-1}-A_{n+1}}{4n}\ , \\ \alpha_n+\beta_n=\dfrac{(n+1)(n-2)A_{n-2}+2nA_n-(n-1)(n+2)A_{n+2}}{8n(n-1)(n+1)}\ , \quad n>1\ . \end{cases}$$

Also $J(1)=0$ gives

(18) $$\beta_1=-\sum_{n=2}^{\infty}n^2\beta_n\ .$$

Now substituting the expansion for $I(x)$ and $J(x)$ in (12) and equating the coefficients of $T_n^{*\prime}(x)$ we find

(19) $$(n+1)A_{n-2}+\{16n(n-1)(n+1)\varepsilon-2n\}A_n+(n-1)A_{n+2}=0\ , \quad n\geq 3$$

where $\varepsilon=1/\lambda$:

Also corresponding to $n=1$,

(20) $$\varepsilon A_1=\frac{A_3-A_1}{16}+\frac{1}{2}(\alpha_1+\beta_1)\ ,$$

Since the solution is symmetrical about $x=1/2$, we get $A_{2n}=0$ for $n\geq 1$. Now replacing n by $2n-1$, the equation (19) becomes

(21) $$nA_{2n-3}+\{32n(n-1)(2n-1)\varepsilon-(2n-1)\}A_{2n-1}+(n-1)A_{2n+1}=0\ , \quad n\geq 2$$

It may be noted that α_1 and β_1 in (20) can be eliminated by employing (16), (17) and (18).

If we now suppose that $f(x)$ is approximated by a polynomial of degree four, then

$$f(x)=A_1T_1^{*\prime}(x)+A_3T_3^{*\prime}(x)+A_5T_5^{*\prime}(x)\ ,$$

where the three equations for A_1, A_3 and A_5 can be obtained from (20) and (21). This can be put in a matrix form as

$$SA=\varepsilon A\ ,$$

where A is the column vector (A_1, A_3, A_5) and S is the matrix

$$\begin{bmatrix} \frac{3}{32} & -\frac{1}{16} & -\frac{1}{96} \\ -\frac{1}{96} & \frac{1}{64} & -\frac{1}{192} \\ 0 & -\frac{1}{320} & \frac{1}{192} \end{bmatrix}$$

The largest eigen-value of this matrix is $\lambda=9.86963$. *Crout* [7] and *Elliott* [1] find the value of λ as 9.87605 and 9.86958 respectively, the analytic solution being 9.86960. The eigen value found here shows an almost closer approximation to the analytic solution like that of *Elliott* and is much better than *Crout*. The magnitudes of the errors being 3×10^{-5}, 2×10^{-5} and 645×10^{-5} respectively.

For the eigenfunction $f(x)$, we note that $f\left(\frac{1}{2}\right)=1$ so that

$$f(x)=0.3609629T_1^{*\prime}(x)-0.0439633T_3^{*\prime}(x)+0.0014294T_5^{*\prime}(x)\ ,$$

The comparison of the functional values obtained by different methods and the analytic solution is shown in Table 3.

Table 3.

x	Exact $\lambda=9.86960$ Sin πx	Crout $\lambda=9.87605$ 4th degree	|error| $\times10^5$	Chebyshev 4th degree Elliott $\lambda=9.86958$	|error| $\times10^5$	Derived Chebyshev 4th degree Present $\lambda=9.86963$	|error| $\times10^5$
0.0, 1.0	0	0	0	0.00058	58	0.00205	205
0.1, 0.9	0.30902	0.30716	186	0.30878	24	0.30862	40
0.2, 0.8	0.58799	0.58716	63	0.58862	83	0.58805	6
0.3, 0.7	0.80902	0.80918	16	0.81000	98	0.80959	57
0.4, 0.6	0.95106	0.95119	13	0.95142	36	0.95130	24
0.5, 0.5	1.00000	1.00000	0	1.00000	0	1.00000	0

From the tabular points it may be observed that the maximum error in this method (205×10^{-5}) is much greater than *Elliott*'s case (98×10^{-5}). But excluding a small region at each of the both ends, the errors at remaining points in the rest of the interval are less than that of *Elliott*.

Taking a sixth degree expansion for $f(x)$ we have the value of $\lambda=9.869604$ and

$$f(x)=0.3608523T_1^{*\prime}(x)-0.0439498T_3^{*\prime}(x)+0.0014294T_5^{*\prime}(x)-0.0000215T_7^{*\prime}(x)\ .$$

The values of the function at different points as compared to *Elliott*'s sixth degree expansion may be seen from the following table:

Table 4.

x	Exact $\lambda=9.86960$ Sin πx	Elliott Chebyshev exp 6th degree $\lambda=9.86966$	$\|error\|$ $\times 10^5$	Present method. Derived Chebyshev exp 6th degree $\lambda=9.869604$	$\|error\|$ $\times 10^5$
0.0, 1.0	0	−0.00004	4	−0.00003	3
0.1, 0.9	0.30902	+0.30906	4	0.30903	1
0.2, 0.8	0.58779	0.58785	6	0.58778	1
0.3, 0.7	0.80902	0.80907	5	0.80902	0
0.4, 0.6	0.95106	0.95107	1	0.95106	0
0.5, 0.5	1.00000	1.00000	0	1.00000	0

4. Collocation points and the eigen-value problem:

In this section we shall be concerned with the choice of collocation points in the eigen-value problem associated with the integral equation having non-degenerate kernel. We have from (9) and (10)

(22)
$$I(x_i)=\sum_{n=1}^{N} a_n\beta_n(x_i)\ ,$$

where

(23)
$$\beta_n(x_i)=\sum_{r=1}^{S}\frac{2n^2}{n^2-(2r-1)^2}b_{2r-1}(x_i)\ ,$$

where n is even and $S=\dfrac{M+1}{2}$ if M is odd, $S=\dfrac{M}{2}$ if M is even and

(24)
$$\beta_n(x_i)=\sum_{r=0}^{S}{}'\frac{2n^2}{n^2-(2r)^2}b_{2r}(x_i)\ ,$$

when n is odd and $S=\dfrac{M-1}{2}$ if M is odd, $S=\dfrac{M}{2}$ if M is even. Equation (11) now reduces to

(25)
$$f(x_i)=F(x_i)+\lambda\sum_{n=1}^{N} a_n\beta_n(x_i)\ ,$$

which can be put in a matrix form as

(26)
$$(T-\lambda\beta)a=F\ ,$$

where T and β are matrices of order N defined by

(27a)
$$T=\begin{bmatrix} 1 & T_2'(x_1) & T_3'(x_1) & \cdots & T_N'(x_1) \\ 1 & T_2'(x_2) & T_3'(x_2) & \cdots & T_N'(x_2) \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 1 & T_2'(x_N) & T_3'(x_N) & \cdots & T_N'(x_N) \end{bmatrix}$$

and

(27b)
$$\beta=\begin{bmatrix} \beta_1(x_1) & \beta_2(x_1) & \cdots & \beta_N(x_1) \\ \beta_1(x_2) & \beta_2(x_2) & \cdots & \beta_N(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ \beta_1(x_N) & \beta_2(x_N) & \cdots & \beta_N(x_N) \end{bmatrix}$$

Also a and F are column vectors of order N given by

(28)
$$\begin{cases} a^t=(a_1, a_2, \cdots a_N) , \\ F^t=(F(x_1), F(x_2), \cdots F(x_N)) . \end{cases}$$

Applying standard methods, the equation (26) can be solved

(29)
$$a=(T-\lambda\beta)^{-1}F .$$

In the notation of equation (26), the homogeneous integral equation viz.,

(30)
$$f(x)-\lambda\int_{-1}^{1} k(x, y)f(y)dy=0 ,$$

reduces to the form

(31)
$$(T-\lambda\beta)a=0 .$$

The determination of eigen-value λ associated with the equation (31) for which the above equation has non-trivial solutions gives the possible eigen-values.

Setting $\lambda=1/\varepsilon$, the equation (31) can be reduced to the more standard from

(32)
$$(T^{-1}\beta)a=\varepsilon a .$$

Hence the solution of the eigen value problem reduces to finding the eigen values of the matrix $T^{-1}\beta$. Now if the points x_i be chosen as the zeros of $T'_{N+1}(x)$ i.e. $x_i=\cos\dfrac{i\pi}{N+1}$, $i=1(1)N$ the matrix T^{-1} can be found explicity. The choice of these collocation points has been made purely for computational convenience. From the orthogonality relation

(33)
$$\begin{cases} \sqrt{1-x_i^2}\sqrt{1-x_j^2}\sum_{r=1}^{N}\dfrac{1}{r^2}T'_r(x_i)T'_r(x_j)=\dfrac{N+1}{2} , & \text{if } i=j \\ =0 . & \text{if } i\neq j \end{cases}$$

and we obtain

(34) $$T^{-1}=\frac{2}{N+1}$$

$$\times\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1/2^2 & 0 & \cdots & 0 \\ 0 & 0 & 1/3^3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1/N^2 \end{bmatrix}\begin{bmatrix} 1 & 1 & \cdots & 1 \\ T_2'(x_1) & T_2'(x_2) & \cdots & T_2'(x_N) \\ \cdots & \cdots & \cdots & \cdots \\ T_N'(x_1) & T_N'(x_2) & \cdots & T_N'(x_N) \end{bmatrix}\begin{bmatrix} 1-x_1^2 & 0 & \cdots & 0 \\ 0 & 1-x_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1-x_N^2 \end{bmatrix}.$$

It is to be noted from the right hand side of (34) that the matrix in the middle is the transposed of T and two other matrices are diagonal matrices whose elements are of the form $\frac{1}{r^2}$ and $1-x_r^2$ respectively, where $r=1,2\cdots N$. Now to obtain the eigen value we compute the matrix $T^{-1}\beta$ and the largest eigen value λ will be determined by the smallest value of ε. If the function $f(x)$ is either even or odd, then the even or odd coefficients a_n respectively are identically zero and so the matrices are reduced in size. The collocation points in these cases are chosen somewhat differently and for the completeness of the discussion we shall state the results explicitly.

(a) Even case:

Let $f(x)$ be an even function, then writing

$$f(x)=\sum_{n=1}^{N+1} a_{2n-1}T'_{2n-1}(x)\ ,$$

$f(x)$ is approximated by a polynomial of degree $2N$ in which there are $N+1$ unknown coefficients. To find out these coefficients we choose

$$x_i=\cos i\pi/2(N+1)\ ,\quad i=1(1)N+1$$

and the following orthogonal property holds good.

(35) $$\left\{\begin{aligned} \sqrt{1-x_i^2}\sqrt{1-x_j^2}\sum_{r=1}^{N+1}\frac{1}{(2r-1)^2}T'_{2r-1}(x_i)T'_{2r-1}(x_j) &= \frac{N+1}{2}\ , \quad \text{if}\ \ i=j \\ &=0\ ,\quad \text{if}\ \ i\neq j \\ &=N+1\ ,\quad \text{if}\ \ i=j=N+1 \end{aligned}\right.$$

The matrix T being of the form

(36) $$T=\begin{bmatrix} T_1'(x_1) & T_3'(x_1) & \cdots T'_{2N-1}(x_1) & T'_{2N+1}(x_1) \\ T_1'(x_2) & T_3'(x_2) & \cdots T'_{2N-1}(x_2) & T'_{2N+1}(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ T_1'(x_{N+1}) & T_3'(x_{N+1}) & \cdots T_{2N-1}(x_{N+1}) & T'_{2N+1}(x_{N+1}) \end{bmatrix},$$

the inverse matrix is given by

$$T^{-1}=\frac{2}{N+1}\begin{bmatrix} 1 & 0 & 0\cdots & 0 \\ 0 & 1/3^2 & 0\cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots\cdots & 1/(2N+1)^2 \end{bmatrix}\begin{bmatrix} T_1'(x_1) & T_1'(x_2) & \cdots & (1/2)T_1'(x_{N+1}) \\ T_3'(x_1) & T_3'(x_2) & \cdots & (1/2)T_3'(x_{N+1}) \\ \cdots & \cdots & \cdots & \cdots \\ T_{2N+1}'(x_1) & T_{2N+1}'(x_2) & \cdots & (1/2)T_{2N+1}'(x_{N+1}) \end{bmatrix}$$

$$\times\begin{bmatrix} 1-x_1^2 & 0 & \cdots & 0 \\ 0 & 1-x_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1-x_{N+1}^2 \end{bmatrix}. \tag{37}$$

(b) Odd Case:

Let $f(x)$ be an odd function of x. Writing

$$f(x)=\sum_{n=1}^{N} a_{2n}T_{2n}'(x)\ ,$$

so that $f(x)$ is approximated by a polynomial of degree $2N-1$ which requires N coefficients to be evaluated.

We choose

$$x_i=\cos \pi i/2(N+1)\ ,\quad i=1(1)N$$

and the follwing orthogonal property holds good.

$$\begin{cases} \sqrt{1-x_i^2}\sqrt{1-x_j^2}\ \sum_{r=1}^{N}\frac{1}{4r^2}\,T_{2r}'(x_i)T_{2r}'(x_j)=\dfrac{N+1}{2}\ , & \text{if } i=j \\ =0\ , & \text{if } i\neq j \end{cases} \tag{38}$$

The matrix T is given by

$$T=\begin{bmatrix} T_2'(x_1) & T_4'(x_1) & \cdots & T_{2N}'(x_1) \\ T_2'(x_2) & T_4'(x_2) & \cdots & T_{2N}'(x_2) \\ \cdots & \cdots & \cdots & \cdots \\ T_2'(x_N) & T_4'(x_N) & \cdots & T_{2N}'(x_N) \end{bmatrix}. \tag{39}$$

With an inverse matrix T^{-1} given by

$$T^{-1}=\frac{2}{N+1}\begin{bmatrix} 1/2^2 & 0 & \cdots & 0 \\ 0 & 1/4^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1/(2N)^2 \end{bmatrix}\begin{bmatrix} T_2'(x_1) & T_2'(x_2) & \cdots & T_2'(x_N) \\ T_4'(x_1) & T_4'(x_2) & \cdots & T_4'(x_N) \\ \cdots & \cdots & \cdots & \cdots \\ T_{2N}'(x_1) & T_{2N}'(x_2) & \cdots & T_{2N}'(x_N) \end{bmatrix}$$

$$
(40) \qquad \times \begin{bmatrix} 1-x_1^2 & 0 & \cdots & 0 \\ 0 & 1-x_2^2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & 1-x_N^2 \end{bmatrix}.
$$

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