A NOTE ON FLUCTUATIONS OF RANDOM WALKS WITHOUT THE FIRST MOMENT

By

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(Received May 12, 1971)

1. Let $X_1, X_2, \cdots$ be a sequence of independent identically distributed real random variables with the common distribution function F. Let $S_0=0$, $S_n=X_1+\cdots+X_n$. Let F^+ and F^- denote the distribution function of X_1^+ and X_1^- respectively, and let $U^-=\sum_{n=0}^{\infty}(F^-)^{n*}$, where F^{n*} is the n-fold convolution of a distribution function F with itself. The purpose of this note is to prove the following theorem.

Theorem. *Assume that* $E|X_1|=\infty$. *Then*

$$P\{S_n>0 \text{ i.o.}\}=0 \quad \text{or} \quad 1\,,$$

according as

$$\int_{-0}^{+\infty} U^-(x)dF^+(x)<\infty \quad or \quad =\infty\,.$$

This theorem can be used to obtain the following result due to *Williamson* [5].

Corollary (*Williamson*). *If* $1-F^-(x)=L(x)x^{-\alpha}$, $x>0$, *where* $0<\alpha<1$ *and* L *varies slowly at* $+\infty$, *then*

$$P\{S_n>0 \text{ i.o.}\}=0 \quad or \quad 1\,,$$

according as

$$\int_{-0}^{+\infty}[1-F^-(x)]^{-1}dF^+(x)<\infty \quad \text{or} \quad =\infty\,.$$

We shall give an example which shows that the assumption on the variation of F^- in the corollary can not be dispensed with.

2. We begin with the proof of the following lemma.

Lemma. *Assume* $F(-0)=0$. *If* f *is a non-negative monotone decreasing function on* $[0,+\infty)$, *then*

$$\sum_{n=0}^{\infty} f(S_n)=\infty \quad or \quad <\infty \qquad \text{w.p.1},$$

according as

$$\int_{-0}^{+\infty} f(x)dU(x)=\infty \quad \textit{or} \quad <\infty\,,$$

where $U=\sum_{0}^{\infty} F^{n*}$.

Proof. Let $Y_n=\sum_{k=0}^{n} f(S_k)$. Then, since $f(S_k)\leq f(S_k-S_j)$ w.p.1 if $j<k$, we have

$$\begin{aligned} EY_n^2 &= \sum_{j,k=0}^{n} Ef(S_j)f(S_k)=\sum_{j=0}^{n} Ef(S_j)^2+2\sum_{j<k} Ef(S_j)f(S_k) \\ &\leq \sum_{j=0}^{n} Ef(S_j)^2+2\sum_{j<k} Ef(S_j)Ef(S_k-S_j)\leq 2[\sum_{j=0}^{n} Ef(S_j)]^2=2(EY_n)^2\,. \end{aligned}$$

By *Kochen* and *Stone*'s generalization of Borel-Cantelli lemma [4], we have

$$P\{\limsup_{n\to\infty} Y_n/EY_n\geq 1\}>0\,.$$

It follows from the *Hewitt-Savage* zero-one law that if

$$\lim_n EY_n=\sum_{n=0}^{\infty} Ef(S_n)=\int_{-0}^{+\infty} f(x)dU(x)=\infty\,, \quad \text{then} \quad \lim_n Y_n=\infty\,, \quad \text{w.p.1.}$$

This proves the lemma.

The proof of our theorem is an application of a result due to *Kesten* [3]. The following statement is a slight modification of his theorem and easily derived from Theorem 5 of [3].

Theorem (*Kesten*). *If* $EX_1^+=\infty$, *then*

$$\limsup_{n\to\infty} X_n^+/\sum_{i=1}^{n} X_i^-=+\infty \quad \text{w.p.1,} \quad \textit{or} \quad =0 \quad \text{w.p.1,}$$

and

$$P\{S_n>0 \text{ i.o.}\}=1 \quad \textit{or} \quad 0\,,$$

according as

$$\limsup_{n\to\infty} X_n^+/\sum_{i=1}^{n} X_i^-=+\infty \quad \textit{or} \quad 0 \quad \text{w.p.1.}$$

Proof of Theorem. If $EX_1^-<\infty$, then $\lim_{x\to+\infty} U^-(x)/x=\left(\int_{-0}^{+\infty} xdF^-(x)\right)^{-1}>0$, and $\lim_{n\to\infty} S_n/n=+\infty$. If $EX_1^+<\infty$, $EX_1^-=\infty$, then $\lim_{n\to\infty} S_n/n=-\infty$ and $\int_{-0}^{+\infty} U^-(x)dF^+(x)$ $=\int_{-0}^{+\infty} [1-F^+(x)]dU^-(x)<\infty$. Hence if $EX_1^+<\infty$ or $EX_1^-<\infty$, then the theorem is trivial, and therefore we may assume that $EX_1^+=EX_1^-=+\infty$. Let $1\leq n_1<n_2<\cdots$ be the successive indices n with $X_n>0$. The random variables

$$V_l=\sum_{n_{l-1}<i<n_l}X_i^- \quad \text{and} \quad W_l=X_{n_l}^+ ,$$

are all independent, all V_l have the same distribution function $G=p\sum_{n=0}^{\infty}(1-p)^n(F^-)^{n*}$, and all W_l have the same distribution function $H=1-p(1-F^+)$. It is obvious that

$$\limsup_{n\to\infty} X_n^+/\sum_{i=1}^{n}X_i^-=\limsup_{k\to\infty} X_{n_k}^+/\sum_{i=1}^{n_k}X_i^-=\limsup_{n\to\infty} W_n/\sum_{l=1}^{n}V_l .$$

It follows from *Kesten*'s theorem that

(1) $$P\{S_n>0 \text{ i.o.}\}=1 \quad \text{or} \quad 0 ,$$

according as

$$P\{W_n>\sum_{l=1}^{n}V_l \text{ i.o.}\}=1 \quad \text{or} \quad 0 .$$

It is obvious that this is equivalent to

$$P\{W_n>\sum_{l=1}^{n}V_l \text{ i.o.}\mid V_1, V_2,\cdots\}=1 \quad \text{or} \quad 0 \quad \text{w.p.l.}$$

Thus by Borel-Cantelli lemma, (1) holds according as

$$\sum_{n=1}^{\infty}P\{W_n>\sum_{l=1}^{n}V_l\mid V_1, V_2,\cdots\}=\infty \quad \text{or} \quad <\infty \quad \text{w.p.l.}$$

It follows from the relation

$$P\{W_n>\sum_{l=1}^{n}V_l\mid V_1, V_2,\cdots\}=1-H(\sum_{l=1}^{n}V_l) , \quad \text{w.p.l,}$$

and from Lemma that (1) holds according as

$$\sum_{n=0}^{\infty}\int_{-0}^{+\infty}[1-H(x)]dG^{n*}(x)=\int_{-0}^{+\infty}[1-H(x)]d(\sum_{n=0}^{\infty}G^{n*}(x))=\infty \quad \text{or} \quad <\infty ,$$

or equivalently

$$\int_{-0}^{+\infty}(\sum_{n=1}^{\infty}G^{n*}(x))dH(x)=\infty \quad \text{or} \quad <\infty .$$

Since $dH(x)=pdF^+(x)$, and $\sum_{n=1}^{\infty}G^{n*}(x)=1+p(1-p)^{-1}U^-(x)$ for $x>0$, this implies the theorem.

Proof of Corollary. If $1-F^-(x)=L(x)x^{-\alpha}$, $x>0$, where $0<\alpha<1$ and L varies slowly at $+\infty$, then it is well-known ([2] p. 446) that

$$\lim_{x\to+\infty}[1-F^-(x)]U^-(x)=(\sin \pi\alpha)/(\pi\alpha) .$$

Hence the corollary follows immediately from Theorem.

Example. Let the common distribution of random variables $X_1, X_2, \cdots$ be such that $P\{X_n=k!\}=c(k!)^{-1}$ for $k\geq 1$, and $P\{X_n=-k!\}=c((k-2)!)^{-1}$ for $k\geq 3$, where $c=(2e-2)^{-1}$. It is easy to verify that

$$(2) \qquad \int_{-0}^{+\infty}[1-F^-(tx)]^{-1}dF^+(x)=\infty \quad \text{or} \quad <\infty\,,$$

according as $t>1$ or $0<t<1$. Let $t'>1$, $0<t''<1$, and let $X'_n=X_n^+-(t')^{-1}X_n^-$, $X''_n=X_n^+-(t'')^{-1}X_n^-$, $S'_0=S''_0=0$, $S'_n=X'_1+\cdots+X'_n$, $S''_n=X''_1+\cdots+X''_n$. It follows from *Kesten*'s theorem that $P\{S'_n>0 \text{ i.o.}\}=P\{S''_n>0 \text{ i.o.}\}=0$ or 1. This fact together with (2) shows that in Corollary we cannot remove the assumption on the variation of $1-F^-$.

Remark. Combined with Theorem 6 of [3] our theorem implies that

$$(3) \qquad \lim_n n^{-1}S_n=-\infty \quad \text{w.p.1},$$

if and only if $E|X_1|=\infty$ and

$$\int_{-0}^{+\infty}U^-(x)dF^+(x)<\infty\,.$$

A sufficient condition for (3) is that there exist constants $0<\alpha<1$, $C>0$ and $x_0>0$ for which $1-F^-(x)\geq Cx^{-\alpha}$ for $x>x_0$ and $\int_0^\infty x^\alpha dF^+(x)<\infty$. In fact $1-F^-(x)\geq Cx^{-\alpha}$ implies the existence of a distribution function G and a constant $M>0$ such that $G(-0)=0$, $1-G(x)=Cx^{-\alpha}$ for $x>M$ and $F^-\leq G$. Since $(F^-)^{n*}\leq G^{n*}$, $n\geq 1$, we have $U^-\leq\sum_n G^{n*}=O(x^\alpha)$ as $x\to\infty$. Thus $\int_0^\infty U^-(x)dF^+(x)<\infty$ follows. This result includes that of *Derman* and *Robbins* [1], which states that (3) holds if for some constants $0<\alpha<\beta<1$, $C>0$ and $x_0<0$, $F(x)\geq C|x|^{-\alpha}$ for $x<x_0$ and $\int_0^\infty x^\beta dF(x)<\infty$.

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