A CHARACTERIZATION OF THE MULTIPLIERS OF BANACH ALGEBRAS

By

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1. Introduction. It is well-known that the multiplier space of a group algebra is isometrically isomorphic to a bounded regular measure algebra (see *Wendel* [10]). *Figà-Talamanca* [2], [3] has investigated the multipliers of p-integrable functions $(1<p<\infty)$ and shown that the space $M(L^p)$ of multipliers of L^p is isometrically isomorphic to the dual space of the linear span $L^p*L^{p'}$, $(1<p<\infty)$, where $L^p=(L^{p'})^*$. Recently the author [6] has proved that the multiplier space of an $A^p(G)$-algebra is isometrically isomorphic to the dual space of a specified linear span under the convolution operator. In this paper we will answer the question of whether the multiplier space of a Banach algebra can be identified with the dual space of some linear span, and describe how to characterize such multiplier spaces. To this end, we commence with some preparations.

2. Preliminaries and the Main Result. Let A be a Banach algebra and B be a Banach space. We say that a (Banach) left (resp. right) A-module is a Banach space B together with a map $(A\times B)\to B$, and we write the value of (a, x) as ax (or xa). The mapping further satisfies the following conditions:

(i) B is an algebraic left map (resp. right) A-module, i.e., for $a, b\in A$ and $x, y\in B$

$$(a+b)x=ax+bx\ ,\quad a(x+y)=ax+ay\ ,\quad a(bx)=(ab)x\ ,$$

(ii) For every $a\in A$ and $x\in B$,

$$\|ax\|_B\leq\|a\|_A\|x\|_B \quad (\text{resp. } \|xa\|_B\leq\|a\|_A\|x\|_B)\ .$$

In the remainder of this note, an A-module always means a left A-module.

Let B be an A-module. The Banach space B is said to be an essential A-module if the linear manifold S spanned by $\{ax;\ a\in A,\ x\in B\}$ is dense in B.

An approximate identity for a Banach algebra A is a net $\{e_i\}_{i\in I}$ (where I is a directed set) of elements of A having the following properties:

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$$e_i a \to a \text{ (left approximate identity),}$$

and

$$a e_i \to a \text{ (right approximate identity),}$$

for any $a \in A$. If there is a constant C such that $\|e_i\|_A \leq C$ for all i, then the approximate identity is said to be bounded.

If the Banach algebra A has a bounded approximate identity, then one can show easily that Banach space B is an essential A-module if and only if there exists $e_k \in A$ such that $e_k v \to v$ ($v e_k \to v$) for $v \in B$. Indeed, if $\{e_k\}$ is a bounded approximate identity for A, i.e., $\|e_k\|_A \leq C$ for some constant $C>0$ and all k, then, when B is essential, for any given $\varepsilon>0$ and $v \in B$, there is a finite sum $\sum_{i=1}^{n} a_i x_i$ with $a_i \in A$ and $x_i \in B$ such that

$$\Big\| \sum_{i=1}^{n} a_i x_i - v \Big\|_B < \varepsilon/2(C+1) .$$

Now by virtue of the existence of an approximate identity for A, there is an index k_0 with the property that

$$\|e_k a_i - a_i\|_A < \varepsilon/2n\|x_i\|_B \quad \text{for} \quad k>k_0 \quad \text{and } 1 \leq i \leq n .$$

Since

$$\begin{aligned} \|e_k v - v\|_B &= \Big\|e_k v - e_k \sum_{i=1}^{n} a_i x_i + e_k \sum_{i=1}^{n} a_i x_i - \sum_{i=1}^{n} a_i x_i + \sum_{i=1}^{n} a_i x_i - v\Big\|_B \\ &\leq \|e_k\|_A \Big\|v - \sum_{i=1}^{n} a_i x_i\Big\|_B + \sum_{i=1}^{n} \|(e_k a_i - a_i) x_i\|_B + \Big\|\sum_{i=1}^{n} a_i x_i - v\Big\|_B \\ &\leq (\|e_k\|_A + 1)\Big\|v - \sum_{i=1}^{n} a_i x_i\Big\|_B + \sum_{i=1}^{n} \|e_k a_i - a_i\|_A \|x_i\|_B \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon . \end{aligned}$$

Conversely, it is immediate that B is an essential A-module.

Let A be a Banach algebra. The bounded linear operator T on A is said to be a left (resp. right) multiplier of A if

$$T(ab) = Ta(b) \quad (\text{resp. } T(ab) = (a)Tb) .$$

If A is a commutative Banach algebra, the operator T on A is linear and continuous whenever

$$Ta(b) = (a)Tb = T(ab) .$$

(cf. Wang [9]).

If B is an A-module, and if the dual space B^* of B is isometrically isomorphic to A, we naturally take

$$\langle x, ab\rangle=\langle bx, a\rangle \quad (\text{resp. } \langle x, ab\rangle=\langle xb, a\rangle)\,,$$

for any $a, b\in A$ and $x\in B$.

Define a subspace of B by

$$S=\{u;\, u=\sum_{i=1}^{\infty} f_i g_i\,, \quad \text{for} \quad f_i\in A \quad \text{and} \quad g_i\in B \quad \text{and} \quad \sum_{i=1}^{\infty}\|f_i\|_A\|g_i\|_B<\infty\}\,,$$

and the function $u\to|||u|||$ by

$$|||u|||=\inf\{\sum_{i=1}^{\infty}\|f_i\|_A\|g_i\|_B;\, u=\sum_{i=1}^{\infty} f_i g_i\in S\}\,,$$

where the infimum is taken over all $f_i\in A$ and $g_i\in B$ such that $u=\sum_{i=1}^{\infty} f_i g_i$ with $\sum_{i=1}^{\infty}\|f_i\|_A\|g_i\|_B<\infty$.

It is easy to see that the function $|||u|||$ is a norm in S. And since

$$\|f_i g_i\|_B\le\|f_i\|_A\|g_i\|_B\,,$$

it follows that

$$\|u\|_B\le|||u||| \quad \text{for any} \quad u=\sum_{i=1}^{\infty} f_i g_i \quad \text{in} \quad S\,.$$

Hence the topology so defined is stronger than the topology induced from B. We will show that S is a Banach space with respect to the norm $|||\cdot|||$.

Proposition 1. *The subspace S of B is a Banach space with respect to the norm $|||\cdot|||$ in which the topology so defined is stronger than the topology induced from B.*

Proof. It remains to show the completeness of S with respect to the norm $|||\cdot|||$, and this can be shown by the same argument, mutatis mutandis, as that which was used in the proof of Theorem 2.4 in *Gaudry* [11].

It is easy to see that A has a bounded approximate identity $\{e_i\}$ and $e_i v\to v$ for any $v\in B$, if and only if S is dense in B, i.e., B is an essential A-module. With this information at hand, we can establish the following main theorem.

Theorem 2. *Let A be a Banach algebra with a bounded approximate identity and B an A-module. If A is isometrically isomorphic to the conjugate space B^* of B, then the conjugate space S^* of a subspace S of B defined above is isometrically isomorphic to the (left) multiplier algebra $M(A)$ of A.*

Note that the Banach algebras $A^p(G)$ need not have a bounded approximate identity (cf. *Lai* [7]). And in *Figa-Talamanca* [2], [3] the space $L^p(G)$ is not a Banach algebra in general.

3. Proof of Theorem. Define the mapping $T\rightarrow\mu$ of $M(A)$ into S^* by

$$\langle u, \mu\rangle=\sum_{i=1}^{\infty}\langle g_i , Tf_i\rangle ,$$

for any $v=\sum_{i=1}^{\infty} f_i g_i$ in A. Thus μ is well-defined. To show this, it suffices to prove that $\langle u, \mu\rangle$ is independent of the choice of the representation for u.

Let $u=\sum_{i=1}^{\infty} f_i g_i=0$. We will show that

$$\langle u, \mu\rangle=\sum_{i=1}^{\infty}\langle g_i , Tf_i\rangle=0 .$$

Let $\{e_\alpha\}$ be a bounded approximate identity for A. Then we have $\|e_\alpha\|_A\leq C$ for some constant C and all α. If $Te_\alpha=h_\alpha$, then

$$\begin{aligned}|\langle g_i , h_\alpha f_i\rangle-\langle g_i , Tf_i\rangle|&\leq\|h_\alpha f_i-Tf_i\|_A\|g_i\|_B\\ &\leq\| T\| \ \|e_\alpha f_i-f_i\|_A\|g_i\|_B\\ &\longrightarrow 0 ,\end{aligned}$$

with the limit being taken over the index α. As

$$u=\sum_{i=1}^{\infty} f_i g_i=0 , \qquad \text{and} \qquad |||u|||\leq \sum_{i=1}^{\infty}\|f_i\|_A\|g_i\|_B<\infty ,$$

we have

$$\sum_{i=1}^{\infty}\langle g_i , h_\alpha f_i\rangle=\sum_{i=1}^{\infty}\langle f_i g_i , h_\alpha\rangle=\langle \sum_{i=1}^{\infty} f_i g_i , h_\alpha\rangle=0 ,$$

for any α. On the other hand,

$$\begin{aligned}|\sum_{i=1}^{\infty}\langle g_i , Tf_i\rangle|&=|\sum_{i=1}^{\infty}\langle g_i , Tf_i\rangle-\sum_{i=1}^{\infty}\langle g_i , h_\alpha f_i\rangle|\\ &\leq|\sum_{i=1}^{N}\langle g_i , Tf_i\rangle-\sum_{i=1}^{N}\langle g_i , h_\alpha f_i\rangle|\\ &\quad+|| T\|(1+C)\sum_{N+1}^{\infty}\|f_i\|_A\|g_i\|_B .\end{aligned}$$

The right hand side of this last inequality can be made arbitrarily small by taking a sufficiently large positive integer N, and then passing to the limit with respect to α. Hence $\sum_{i=1}^{\infty}\langle g_i , Tf_i\rangle=0$ and this proves that μ is well-defined.

The mapping $T\rightarrow\mu$ defined above is obviously injective. We will show that the mapping of $M(A)$ into S^* is also surjective. Thus, let $\mu\in S^*$ and for an arbitrary fixed element $f\in A$, define the linear functional

$$g\rightarrow\langle fg, \mu\rangle=\langle g, t_f\rangle .$$

for $g\in B$. Then, as $\mu\in S^*$,

$$|\langle fg, \mu\rangle|\leq\|\mu\|\ |||fg|||\leq\|\mu\|\ \|f\|_A\|g\|_B\ ,$$

or

$$|\langle g, t_f\rangle|\leq\|\mu\|\ \|f\|_A\|g\|_B\ .$$

Hence t_f defines an element of B^* such that

$$\|t_f\|\leq\|\mu\|\ \|f\|_A\ .$$

By assumption, $A\cong B^*$, so there is a unique element, say $Tf\in A$, such that

$$\langle g, Tf\rangle=\langle fg, \mu\rangle=\langle g, t_f\rangle\ ,$$

and

$$\|Tf\|_A=\|t_f\|\leq\|\mu\|\ \|f\|_A\ .$$

Therefore T is a bounded transformation of A itself which carries f to Tf defined as above. It is obvious that T is a linear mapping. We have to show this bounded linear mapping T is precisely a multiplier of A. Indeed, for $f, h\in A$ and $g\in B$,

$$\langle(fh)g, \mu\rangle=\langle g, T(fh)\rangle\ ,$$

$$\langle f(hg), \mu\rangle=\langle hg, Tf\rangle=\langle g, Tf(h)\rangle\ .$$

Thus $T(fh)=Tf(h)$, and this shows that T is a (left) multiplier of A.

Finally, we prove that the mapping $T\rightarrow\mu$ is an isometry.

Since

$$|\langle u, \mu\rangle|=|\sum_{i=1}^{\infty}\langle g_i, Tf_i\rangle|\leq\sum_{i=1}^{\infty}\|Tf_i\|_A\|g_i\|_B$$
$$\leq\|T\|\sum_{i=1}^{\infty}\|f_i\|_A\|g_i\|_B\ ,$$

it follows that

$$|\langle u, \mu\rangle|\leq\|T\|\ |||u|||\ .$$

Therefore

$$\|\mu\|\leq\|T\|\ .$$

On the other hand, as $A\cong B^*$, we see that

$$\|T\|=\sup_{\|f\|_A\leq1}\|Tf\|_A=\sup_{\substack{\|g\|_B\leq1\\ \|f\|_A\leq1}}|\langle g, Tf\rangle|\leq\sup_{\|f\|_A\|g\|_B\leq1}|\langle fg, \mu\rangle|\leq\|\mu\|\ .$$

Hence $\|\mu\|=\|T\|$. The proof is now complete. Q.E.D.

Remark. For the right multiplier algebra, one can obtain a similar theorem in an analogous way. If A is a commutative Banach algebra without order, then

A is strictiy dense in $M(A)$ if and only if A has an approximate identity. Thus, when B is a two-sided essential A-module with a bounded approximate identity for A such that $B^* \cong A$, then S is a dense subset of B, and the following relation holds (cf. *Wang* [9])

$$\begin{array}{ccc} A & \subset & M(A) \\ \| \wr & & \| \wr \\ B^* & \subset & S^* \end{array}$$

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REFERENCES

[1] Hisashi Choda and Masahiro Nakamura, *Multipliers of Banach algebras*, Proc. Japan Acad. 38 (1962), 93-94.

[2] A. Figa-Talamanca, *Translation invariant operators in L^p*. Duke Math. J. 32 (1965), 495-501.

[3] A. Figa-Talamanca, *Multipliers of p-integrable functions*, Bull. Amer. Math. Soc. 70 (1964), 666-669.

[4] A. Figa-Talamanca and G. I. Gaudry, *Density and representation theorem for multipliers of type (p, q)*, J. Austr. Math. Soc. (1967), 1-7.

[5] E. Hewitt, *The ranges of certain convolution operators*, Math. Scand. 15 (1964), 147-155.

[6] H. C. Lai, *On the multipliers of $A^p(G)$-algebras*, Tohoku Math. J. 23 (1971), 641-662.

[7] H. C. Lai, *On some properties of $A^p(G)$-algebras*, Proc. Japan Acad. 45 (1969), 572-576.

[8] L. Mate, *The Arens product and multiplier operators*, Studia Math. 28 (1967) 227-234.

[9] J. K. Wang, *Multipliers of commutative Banach algebras*, Pacific J. Math. 11 (1961) 1131-1149.

[10] J. G. Wendel, *Left centralizers and isomorphisms of group algebras*, Pacific J. Math. 2 (1952), 251-261.

[11] G. I. Gaudry, *Quasimeasures and operators commuting with convolution*, Pacific J. Math. 18 (1966), 461-476.

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