PARTITION SPACES

By

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(Received November 7, 1970)

Introduction

It is well-known that the method of taking quotient is very useful in solving problems in mathematics. It seems that this method is yielding more fruitful results in algebra than in general topology. The reason would probably be that in defining an algebraic quotient, the decomposition of the underlying set should be compatible with the composition laws of the algebraic structure in question, whereas in the topological case, any decomposition of the underlying set yields a quotient space.

A comparison between the algebraic quotient and the topological quotient leads us to consider a restrictive type of topological quotients which is, in some way, similar to the algebraic quotients. We define: a quotient space X/R of a topological space X is called a *partition space* of X if the following requirement is satisfied:

For any $A \subset X$ and xRy, $x \in A^0$ if and only if $y \in A^0$.

It turns out that the notion of partition space has been implicit in different contexts.

Example 1. *T_0 partition spaces.*

Let X be an arbitrary topological space, and R the equivalence relation on X such that xRy if and only if $x \in y^-$ and $y \in x^-$. It is easy to see that this quotient space $X^\pi = X/R$ enjoys the following pleasant properties:

(i) *X^π is a T_0 space.*

(ii) *X^π is a partition space of X.*

(iii) *The topology of X^π is lattice-isomorphic to the topology of X.*

In passing over to the T_0 partition space X^π, we are in a position to reduce some problems on an arbitrary topological space X to problems on a T_0 space X^π leaving the lattice of the topology of X essentially intact. Using this method, *H. Gaifman* [6] and *A. K. Steiner* [15] obtained results on the existence of a complement or a principal complement of a topology of a topological space.

Example 2. *Compact Hausdorff partition spaces.*

The concept of partition spaces was considered by *J. Hardy* and *H. E. Lacey* in 1968 [10]. Their paper is concerned with the extension of regular Borel measures defined on the Borel sets generated by subtopologies of a compact regular space. *J. Hardy* and *H. E. Lacey*'s difinition is as follows:

Let X be a compact regular space. For each point $x \in X$, let N_x be the set of all points $y \in X$ such that for each open set U, $y \in U$ if and only if $x \in U$. Consider $Y = \{N_x : x \in X\}$. Define $f : X \to Y$ by $f(x) = N_x$ for every $x \in X$, and give to Y the largest topology for which f is continuous.

It is proved that Y is a compact Hausdorff partition space of X. The authors claim that the concept of a partition space plays a central role in the development of their work.

Example 3. *The space L^2.*

In real analysis and functional analysis, we study the space $\mathscr{L}^2$ of all measurable functions f which are square-integrable (in Lebesgue's sense). $\mathscr{L}^2$ is given the strong topology induced by the norm

$$\|f\| = \left(\int |f|^2 d_\mu\right)^{1/2} .$$

If we identity f and g of $\mathscr{L}^2$ when the set $\{x : f(x) \neq g(x)\}$ is of measure zero, then we obtain the space L^2 which turns out to be a partition space of $\mathscr{L}^2$.

Example 4. *Associated metric space.*

Let (X, d) be a pseudo-metric space. Consider the equivalence relation R such that xRy if and only if $d(x, y) = 0$. Then the quotient spoce X/R is a metric space with respect to the metric $d^*([x], [y]) = d(x, y)$. Here again X/R is a partition space of X.

From the above examples, one sees that partition spaces appear explicitly or implicitly in a number of ways in mathematics. It seems to us that a systematic study of them may be worthwhile.

The general theory of partition spaces is developed in Chapter I, where we define a covariant functor π of the category of topological spaces and continuous functions into the subcategory of T_0 spaces. The functor π carries imbeddings into imbeddings and homeomorphisms into homeomorphisms.

In Chapter II, we study a new classification of topological spaces by π-equivalence. In relation to the well-known classifications we have:

$$\text{homeomorphic} \Longrightarrow \pi\text{-equivalent} \Longrightarrow \begin{cases} \text{lattice-equivalent} \\ \text{homotopically equivalent} \end{cases}$$

A study of π-invariant properties is included.

Chapter III is devoted to a study of of some separation axioms. There we introduce the t_1, t_2 and t_4 axioms and prove that

$$\begin{array}{ccccccccccc} T_4 & \Longrightarrow & \text{Tychonoff} & \Longrightarrow & T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ t_4 & \Longrightarrow & \begin{array}{c}\text{completely}\\ \text{regular}\end{array} & \Longrightarrow & \text{regular} & \Longrightarrow & t_2 & \Longrightarrow & t_1 & & \end{array}$$

The following principle:

Suppose P and Q are π-invariant properties. If the implication $P \Longrightarrow Q$ is true for all T_i spaces, then the implication is also true for all t_i spaces,

is used to generalize results in T_i spaces. Among others we prove that

A topological space is completely regular if and only if it possesses a normal base,

and

Every compact pseudo-metric space is supercompact.

Many of the well-known compactification theories (e.g. the *Stone-Čech* compactification and *Wallman-Frink* compactification) apply only to Hausdorff spaces. The Hausdorff separation axiom is thought to be necessary to ensure some form of uniqueness of compactification. In the last chapter we generalize *Tychonoff*'s theorem:

A topological space is homeomorphic to a subspace of a compact Hausdorff space if and only if it is a completely regular T_1 space,

into the following:

A topological space is homeomorphic to a subspace of a compact t_2 space if and only if it is a completely regular space.

We further generalize the methods of Stone-Čech compactification and Wallman-Frink compactification to cater for completely regular spaces.

This paper is based on a doctoral thesis of the University of Hong Kong. The author wishes to thank his supervisors, Professor *Y. C. Wong* and Dr. *K. T. Leung*, for their kind guidance.

CHAPTER I. Partition Spaces

1. Definition and Characterization of Partition Spaces

Let X be a topological space and R an equivalence relation on X. The quotient space X/R is said to be a *partition space* of X if the following condition is satisfied:

For each $A \subset X$, *if* $x \in A^0$ *and* xRy *then* $y \in A^0$, *where* A^0 *denotes the interior of the set* A *in* X.

In other words, X/R is a partition space if and only if R-related points of X belong to the same open sets of X. By duality, X/R is a partition space if and only if R-related points of X belong to the same open sets of X. By duality, X/R is a partition space if and only if R-related points of X belong to the same closed sets of X.

The existence of a partition space of any topological space is evident. In fact, for any topological space X, X itself is always a partition space of X. In general, a topological space may have more than one partition space. For example, if X is an indiscrete space containing more than one point, then every quotient space X/R is a partition space of X. The condition for uniqueness of of partition spaces of a topological space is given in the following theorem:

THEOREM 1.1. *A topological space has a unique* (*up to a homeomorphism*) *partition space if and only if it is a* T_0 *space.*

PROOF. *Necessity*: We have seen in Example 1 (Introduction) that any space X has a T_0 partition space X^π. On the other hand, X itself is a partition of X, therefore if X has unique partition space, then X is homeomorphic to X^π, and hence X has to be a T_0 space.

Sufficiency: Let X be a T_0 space. We shall show that the only partition space of X is the trivial partition space X. Let X/R be a partition space of X. Suppose R is not the diagonal of $X \times X$, then we can find $x \neq y$ in X such that xRy. Since X is T_0, we have, say, an open neighbourhood U of x such that $y \notin U$. This contradicts the assumption that X/R is a partition space of X.

According to the above theorem, the method of taking partition spaces is useful only for non-T_0 spaces. As a matter of fact, the usefulness of the method of taking partition spaces bases on the fact that every topological space has one and only one T_0 partition space (see §2, Corollary 1.6 of this chapter).

We shall present here a characterization of partition spaces, which will be made use of frequently.

THEOREM 1.2. *Let* X/R *be a quotient space of a topological space* X *and* p *the natural projection from* X *onto* X/R. X/R *is a partition space of* X *if and only if for every open set* G *in* X,

$$G = p^{-1}(p(G)) .$$

PROOF. Suppose X/R is a partition space of X and G an open set of X. If $y \in p^{-1}(p(G))$, then there is some $x \in G$ such that $p(x) = p(y)$ i.e. xRy. Therefore $y \in G$, by definition of partition space. Thus $G \supset p^{-1}(p(G))$. On the other hand, it is generally true that $G \subset p^{-1}(p(G))$, therefore $G = p^{-1}(p(G))$ for every open set G.

Conversely, suppose $G = p^{-1}(p(G))$ for every open set G of X. For any $A \subset X$, let $x \in A^0$ and xRy. Since $x \in A^0$ and $p(x) = p(y)$, $p(y) \in p(A^0)$ and hence $y \in p^{-1}(p(A^0)) = A^0$. Therefore X/R is a partition space of X.

By duality, the above theorem can be restated as follows:

COROLLARY 1.3. *X/R is a partition space of X if and only if for every closed set F in X, $F = p^{-1}(p(F))$.*

A further consequence of Theorem 1.2 is the following useful corollary.

COROLLARY 1.4. *If X/R is a partition space of X, then the natural projection $p : X \rightarrow X/R$ is an open and closed map.*

We note that the above property of p is not sufficient for a quotient space to be a partition space. Indeed, if Y is a discrete space with more than one point, then Y has more than one quotient space and for each quotient space of Y the natural projection is an open and closed map. However being a T_1 space, Y has only one partition space.

2. The Partial Ordering on Partition Spaces.

Let X/R_1 and X/R_2 be two quotient spaces of X. We shall say that X/R_1 is larger than X/R_2 and write $X/R_1 \geqslant X/R_2$ (or X/R_2 is smaller than X/R_1 and write $X/R_2 \leqslant X/R_1$) if for all x, y in X, xR_1y implies xR_2y. Clearly, $\geqslant$ is an ordering and the trivial quotient spaces $\{X\}$ and X are the smallest and the largest element of the set $Q(X)$ of all quotient spaces of X. We shall show that every family of quotient spaces of X has a least upper bound as well as a greatest lower bound. Consequently the set $Q(X)$ of all quotient spaces of a topological space X is a complete lattice with respect to this ordering.

Let $\{X/R_\alpha\}_{\alpha \in A}$ be a family of quotient spaces of X. We define equivalence relations R and R' on X by:

> *xRy if and only if $xR_\alpha y$ for all $\alpha \in A$. $xR'y$ if and only if there exist a positive integer n and $\alpha_1, \cdots, \alpha_{n+1} \in A$ and $t_1, \cdots, t_n \in X$ such that $xR_{\alpha_1}t_1$, $t_1R_{\alpha_2}t_2, \cdots, t_{n-1}R_{\alpha_n}t_n$ and $t_nR_{\alpha_{n+1}}y$.*

Then X/R and X/R' are the least upper bound and the greatest lower bound of the family $\{X/R_\alpha\}_{\alpha \in A}$ respectively.

The ordering on $Q(X)$ is essentially an ordering of the equivalence relations in the set X and is unrelated to the topology of X or to the topologies of the quotient spaces. However interesting results can be obtained when we consider the induced ordering on the subset of all partition spaces.

THEOREM 1.5. *The set of all partition spaces of a topological space X forms a complete sub-lattice of the lattice $Q(X)$ of all quotient spaces of X. Furthermore, the smallest partition space is the largest T_0 quotint space of X.*

PROOF. For the first part, it suffices to verify that the lattice sum X/R as well as the lattice product X/R' of a family $\{X/R_\alpha\}_{\alpha\in A}$ of partition spaces of X are partition spaces of X. By definition, xRy if and only if $xR_\alpha y$ for all R_α. Therefore R-related points of X belong to the same open sets of X and X/R is a partition space of X. Suppose $xR'y$. Then by definition of R', $xR_{\alpha_1}t_1, \cdots, t_nR_{\alpha_{n+1}}y$ for some $t_i\in X$ and $\alpha_i\in A$. Therefore $x, t_1, \cdots, t_n$ and y all belong to the same open sets of X and hence X/R' is a partition space.

Let S be the equivalence relation on X, defined as the following:

xSy if and only if $x\in y^-$ and $y\in x^-$, where x^- stands for the closure of the singleton $\{x\}$.

We shall show that X/S, with the quotient topology, is the smallest partition space as well as the largest T_0 quotient space of X. We have seen in Example 1 (Introduction) that X/S is a T_0 partition space of X. Suppose X/Q is a partition space of X and xQy. Since $y\in y^-$, $x\in x^-$ and X/Q is a partition space, we have $x\in y^-$ and $y\in x^-$ i.e., xSy. Since x, y are arbitrary, we have $X/Q\geqslant X/S$.

It remains to prove that X/S is the largest T_0 quotient space of X. Let X/T be a T_0 quotient space of X. We are going to show that $X/S\geqslant X/T$. Let $p: X\to X/T$ be the natural projection and suppose xSy. We shall show that $p(x)=p(y)$. Assume $p(x)\neq p(y)$. Since X/T is a T_0 space, there is, say, an open neighbourhood U of $p(x)$ which does not contain $p(y)$. $p^{-1}(U)$ is an open neighbourhood of x which does not contain y. We have then $x\notin y^-$, contradicting xSy.

COROLLARY 1.6. *If X is a topological space, then X has one and only one T_0 partition space denoted henceforth by X^π.*

COROLLARY 1.7. *If X is a T_0 space, then $X^\pi=X$ and all partition spaces of X are equal to itself.*

The following theorem is a characterization of the T_0 partition space of X.

THEOREM 1.8. *Let X/R be a quotient space of a topological space X and p the natural projection. $X/R=X^\pi$* (i.e., *X/R is the T_0 space of X*) *if and only if for all x, $y\in X$, the following statements are equivalent*:

(i) $p(x)=p(y)$;

(ii) *For all* $A \subset X$, $x \in A^-$ *if and only if* $y \in A^-$;

(iii) *For all* $A \subset X$, $x \in A^0$ *if and only if* $y \in A^0$.

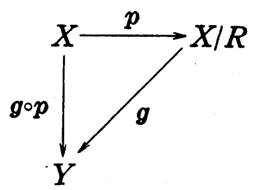
PROOF. *Necessity*: It is obvious that (ii)$\Longleftrightarrow$(iii) holds in general. Suppose X/R is the T_0 partition space of X. By definition, we have (i)$\Longleftrightarrow$(iii). We now show (iii)$\Longleftrightarrow$(i). Assume $p(x) \neq p(y)$. Since X/R is T_0, there is an open set G in X/R such that, say, $p(x) \in G$ but $p(y) \notin G$. $p^{-1}(G)$ is then an open set in X so that $x \in p^{-1}(G)$ but $y \notin p^{-1}(G)$, contradicting (iii).

Sufficiency: That X/R is a partition space follows from (i)$\Longleftrightarrow$(iii). We now prove that X/R is a T_0 space. Let $p(x) \neq p(y)$. From (iii)$\Longleftrightarrow$(i), an open set G in X exists such that, say, $x \in G$ but $y \notin G$. By Theorem 1.2, $G = p^{-1}(p(G))$. Therefore, $p(G)$ is open in X/R and $p(x) \in p(G)$ but $p(y) \notin p(G)$.

3. The Induced Maps on Partition Spaces.

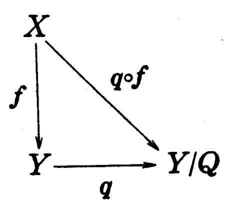
The main purpose of this section is to prove that, for any topological spaces X and Y, the set of all continuous functions of X into Y is mapped in a natural way onto the set of all continuous function of X^π into Y^π.

Let X/R be a quotient space of a topological space X and $g: X/R \to Y$ a function of X/R into a topological space Y. It is well-known that g is continuous if and only if $g \circ p$ is continuous.



For partition spaces, we have other useful lemmas.

LEMMA 1.9. *Let* Y/Q *be a partition space of a topological space* Y *and* q *the natural projection of* Y *onto* Y/Q. *A function* f *from a topological space* X *into* Y *is continuous if and only if* $q \circ f$ *is continuous.*



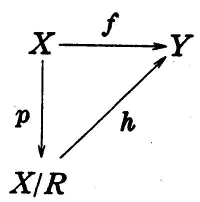
PROOF. The necessity follows from the continuity of the natural projection q. We now prove the sufficiency. Let G be an open set in Y. By Theorem

1.2, $G=q^{-1}(q(G))$. Therefore we have

$$f^{-1}(G)=f^{-1}(q^{-1}(q(G)))=(q\circ f)^{-1}(q(G)) .$$

Since q is open (Corollary 1.4) and $q\circ f$ is continuous (assumption), $f^{-1}(G)$ is open. f is therefore continuous.

LEMMA 1.10. *Let X/R be a partition space of a topological space X and p the natural projection of X onto X/R. For every continuous function $f: X\rightarrow Y$ of X into a T_0 space Y, there exists a unique continuons function $h: X/R\rightarrow Y$ such that the diagram*

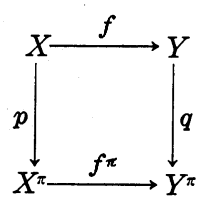


is commutative.

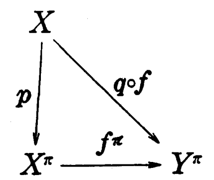
PROOF. The uniqueness of h is obvious. To prove the existence of h, it is enough to show that $x_1, x_2\in X$, if $p(x_1)=p(x_2)$ then $f(x_1)=f(x_2)$. Suppose this is not the case, then there are $x_1, x_2\in X$ such that $p(x_1)=p(x_2)$ and $f(x_1)\neq f(x_2)$. Since Y is T_0, we have, say, an open neighbourhood G of $f(x_1)$ in Y which does not contain $f(x_2)$. By continuity $f^{-1}(G)$ is open. By construction, $x_1\in f^{-1}(G)$ and $x_2\notin f^{-1}(G)$. As X/R is a partition space of X we should have $x_2\in f^{-1}(G)$. This is a contradition. The continuity of h follows from the commutativity of the diagram and that X/R is a quotient space.

Now we are in a position to present the first main theorem as follows:

THEOREM 1.11. *Let X^π, Y^π be the T_0 partition spaces of X, Y and p, q the corresponding natural projections. For any continuous function $f: X\rightarrow Y$, there exists a unique continuous function $f^\pi: X^\pi\rightarrow Y^\pi$ such that the following diagram is commutative.*



PROOF. By Lemma 1.9, $q\circ f$ is continuous. Therefore by Lemma 1.10, there exists a unique continuous function $f^\pi: X^\pi\rightarrow Y^\pi$ such that the diagrem



is commutative, and the theorem follows.

It turns out that the induced function f^π inherits some properties of f.

THEOREM 1.12. *If f is an imbedding then f^π is also an imbedding.*

PROOF. We first show that f^π is injective. Suppose the contrary is true. Then there are $x,\ x' \in X$, such that $f^\pi(p(x))=f^\pi(p(x'))$ and $p(x)\neq p(x')$. From the equation we have $q(f(x))=q(f(x'))$ since $f^\pi p=qf$. From the inequality of elements of the T_0 partition space X^π, there exists an open neighbourhood U of x, which does not contain x'. Since f is an imbedding, this implies the existence of an open set in Y, which contains $f(x)$ but not $f(x')$, contradicting $q(f(x))=q(f(x'))$ in the partition space Y^π. This shows that f^π is injective. To prove that f^π is an imbedding, it is now enough to show that for any open set G in X^π, $f^\pi(G)$ is open in $f^\pi(X^\pi)$. Let $U=p^{-1}(G)$. As $qf=f^\pi p$ it is equivalent to show that $q(f(U))$ is open in $q(f(X))$. Since f is an imbedding, $f(U)$ is open in $f(X)$, i.e., $V\cap f(X)=f(U)$ for an open set V of Y. Since q is open (Corollary 1.4), $q(V)$ is an open set in Y. We shall show that $q(f(U))=q(V)\cap q(f(X))$. Clearly $q(f(U))= q(V\cap f(X))\subset q(V)\cap q(f(X))$. It remains to show that $q(V\cap f(X))\supset q(V)\cap q(f(X))$. Let $z\in q(V)\cap q(f(X))$. Then $z=q(f(x))$ for some $x\in X$. On the other hand from $q(f(x))\in q(V)$ we get $f(x)\in q^{-1}(q(V))=V$. Therefore $z\in q(V\cap f(X))$. The theorem is proved.

COROLLARY 1.13. *If f is a homeomorphim from X onto Y, then f^π is a homeomorphism from X^π onto Y^π.*

PROOF. Making use of the above theorem, we need only to prove that the surjectivity of f implies the surjectivity of f^π. But it follows from the commutativity of the diagram in Theorem 1.11 immediately.

Remark. We note here that in general f^π does not inherit injectivity. Indeed if X and Y are the discrete and the indiscrete space of one and the same set S respectively, then the identity map i of S is a continuous injection from X into Y. We also see that X^π is homeomorphic to X and Y^π is just a singleton; therefore if S has more than one element, then i^π is not injective.

It is seen in Theorem 1.11 that every continuous function f of X into Y gives rise to a continuous function f^π of X^π into Y^π. We now show in the following second main theorem that every continuous function g of X^π into Y^π can be obtained in this way.

THEOREM 1.14. *For every continuous function* $g : X^\pi \to Y^\pi$*, there is a continuous function* $f : X \to Y$ *such that* $g = f^\pi$.

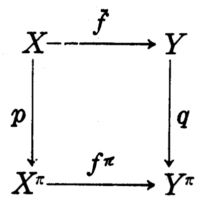
PROOF. Consider the composite function $g \circ p$, which is a continuous function of X into Y. We define a function f as follows.

Let ϕ be a choice function of the decomposition Y^π of Y, i.e., a mapping $\phi : Y^\pi \to Y$ such that $q \circ \phi$ is the identity of Y^π. Let $f = \phi \circ g \circ p$. Then $q \circ f = g \circ p$. Since $g \circ p$ is continuous, f is a continuous function of X into Y, by Lemma 1.9. By uniqueness, we have $f^\pi = g$.

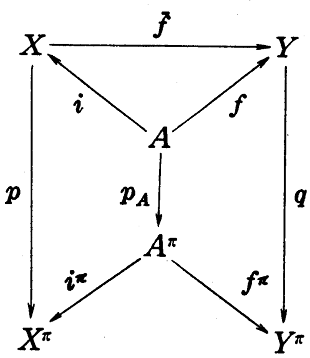
Let A be a subspace of X and i the inclusion map of A into X. Then i^π is an imbedding of A^π into X^π. Our results so far enable us to reduce an extension problem on continuous function between arbitrary topological spaces to one on continuous function between T_0 spaces.

THEOREM 1.15. *Let A be a subspace of a topological space X and $i : A \to X$ the inclusion. Let Y be an arbitrary topological space. Then a continuous function $f : A \to Y$ admits an extension over X if and only if the continuous function $f^\pi : A^\pi \to Y^\pi$ admits an extension over X^π.*[1]

PROOF. *Necessity*: Let $\bar{f} : X \to Y$ be a continuous extension of f, i.e., $f = \bar{f} \circ i$. Using notations of Theorem 1.11 we obtain commutative diagrams:

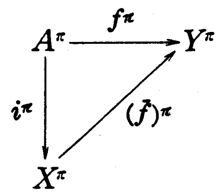


and



1) By Theorem 1.12, corresponding to the inclusion map $i : A \to X$, $i^\pi : A^\pi \to X^\pi$ is an imbedding. By an extension $\bar{f}^\pi$ of f^π here we mean $\bar{f}^\pi : X^\pi \to Y^\pi$ and $\bar{f}^\pi \circ i^\pi = f^\pi$.

Chasing the diagrams, we see that

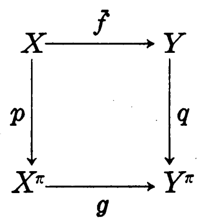


is commutative; therefore $(\bar{f})^\pi$ is an extension of f^π over X^π.

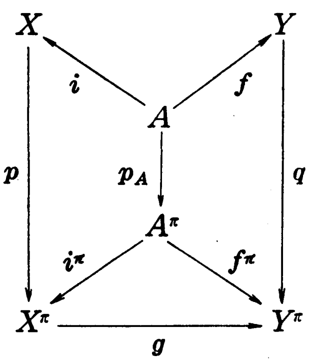
Sufficiency: Suppose f^π admits an extension $g: X^\pi \to Y^\pi$, i.e., $f^\pi = g \circ i^\pi$. Take a choice function $\varphi : Y^\pi \to Y$ as in the proof of Theorem 1.14. Define a map $\bar{f}: X \to Y$ by putting

$$\bar{f}(x) = \begin{cases} f(x) & \text{if} \quad x \in A\,, \\ \varphi \circ g \circ p(x) & \text{if} \quad x \in X \backslash A\,. \end{cases}$$

Clearly $f = \bar{f} \circ i$. It remains to show that $\bar{f}$ is continuous. By Lemma 1.9, it suffices to show commutativity of the diagram



Now commutativity of the following diagram



qualifies the following computations:

If $x \in X \backslash A$, then

$$q(\bar{f}(x)) = q(\varphi(g(p(x)))) = g(p(x))\,.$$

If $x \in A$, then

$$q(\bar{f}(x)) = q(f(x)) = f^\pi(p_A(x)) = g(i^\pi(p_A(x))) = g(p(i(x))) = g(p(x))\,.$$

Therefore $q\circ \tilde{f} = g\circ p$.

4. The Covariant Functor π.

To conclude this chapter, we formalize some of the results between a topological space X and its T_0 partition space X^π in terms of a covariant functor. We denote by $\mathscr{T}$ the category of all topological space with continuous functions as morphisms and by $\mathscr{T}_0$ the subcategory of all T_0 topological spaces. For every object X of $\mathscr{T}$ we put $\pi(X)=X^\pi$, the unique T_0 partition space of X; and for every morphism $f: X\rightarrow Y$ of $\mathscr{T}$ we put $\pi(f)=f^\pi$, the unique continuous function such that

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X^\pi & \xrightarrow{\ f^\pi\ } & Y^\pi \end{array}$$

is commutative. It is easy to verify that

(i) $\pi(i_X)=i_{X^\pi}$ *where* i_X *and* i_{X^π} *are the identity mappings of* X *and* X^π *respectively, and*

(ii) $\pi(f\circ g)=\pi(f)\circ\pi(g)$.

Ie other words, $\pi: \mathscr{T}\rightarrow\mathscr{T}_0$ is a covariant functor of categories. Besides the results of the previous sections, the functor also has properties

(iii) *The restriction of* π *to the subcategory* $\mathscr{T}_0$ *is the identity functor of* $\mathscr{T}_0$, *and*

(iv) $\pi^2=\pi$.

CHAPTER II. π-equivalence of Topological Spaces

In this chapter, we shall discuss a new classification of topological spaces. This new classification is coarser than the classifications by lattice-equivalence and by homotopy equivalence.

1. π-equivalence.

Making use of the functor π of the category $\mathscr{T}$ of all topological spaces into the category $\mathscr{T}_0$ of all T_0 spaces, we can obtain, in a natural way, a classification of objects of $\mathscr{T}$ from a classification of objects $\mathscr{T}_0$. We define

Two topological spaces X and Y are said to be *π-equivalent* if their $\mathscr{T}_0$

partition spaces X^π and Y^π are homeomorphic.

Clearly, π-equivalence is an equivalence relation on objects of $\mathscr{T}$, which in turn gives rise to a classification of topological spaces. It follows from results of the last chapter that X and Y are π-equivalent if and only if there exists a continuous function $f: X\to Y$ such that $f^\pi: X^\pi\to Y^\pi$ is a homeomorphism.

2. Lattice-equivalence and Homotopy Equivalence.

Clearly homeomorphic topological spaces are π-equivalent; consequently the classification by homeomorphism is finer than that by π-equivalence.

After *W. J. Thron* [20] we say that two topological spaces X with topology τ and Y with topology σ are *lattice-equivalent* if there exists a lattice-isomorphism between τ and σ. It is easy to see that lattice-equivalence is again an equivalence relation on objects of $\mathscr{T}$, and that homeomorphic topological spaces are lattice-equivalent. Between π-equivalence and lattice-equivalence we have the following:

THEOREM 2.1. *If X and Y are π-equivalent topological spaces, then X and Y are lattice-equivalent.*

PROOF. It follows from definition that X^π and Y^π are homeomorphic and hence lattice-equivalent. But X and X^π are lattice-equivalent since $p^{-1}(p(G))=G$ for open set G of X and similarly Y and Y^π are lattice-equivalent. Therefore X and Y are lattice-equivalent by transitivity.

From this theorem we see that the classification by π-equivalence is finer than that by lattice-equivalence.

We recall that two continuous functions $f, g: X\to Y$ are homotopic if there exists a continuous function $H: X\times[0,1]\to Y$ such that $H(x,0)=f(x)$ and $H(x,1)=g(x)$ for every $x\in X$. The topological spaces X and Y are said to be homotopically equivalent if there exist $f: X\to Y$ and $g: Y\to X$ such that $f\circ g$ and i_Y are homotopic, and $g\circ f$ and i_X are homotopic. Clearly homotopy equivalence is an equivalence relation on objects of $\mathscr{T}$, and homeomorphic topological spaces are homotopically equivalent. We now prove that the classification by π-equivalence is finer than that by homotopy equivalence.

THEOREM 2.2. *If X and Y are π-equivalent, then X and Y are homotopically equivalent.*

PROOF. It follows from theorem 1.14 and the assumption that X^π and Y^π are homeomorphic that there exist continuous functions $f: X\to Y$ and $g: Y\to X$, such that $f^\pi\circ g^\pi$ and $g^\pi\circ f^\pi$ are the identity maps of Y^π and X^π respectively. In particular, $p\circ g\circ f=p$ and $q\circ f\circ g=q$ where $p: X\to X^\pi$ and $q: Y\to Y^\pi$ are the natural projections. We define $H: X\times[0,1]\to X$ by putting

$$H(x, t)=\begin{cases} g(f(x)) & \text{if } t\neq 1\,, \\ x & \text{if } t=1\,. \end{cases}$$

Then $p\circ H=p\circ h$ where $h: X\times[0, 1]\rightarrow X$ is the projection. Therefore H is continuous by Lemma 1.9. Clearly $H(x, 0)=(g\circ f)(x)$ and $H(x, 1)=x$, therefore $g\circ f$ and i_X are homotopic. Similarly we prove $f\circ g$ and i_Y are homotopic.

To summarise, we have for topological spaces X and Y,

"X and Y are homeomorphic" $\Longrightarrow$ "X and Y are π-equivalent"

$\Longrightarrow \begin{cases} \text{"}X \text{ and } Y \text{ are lattice-equivalent" and} \\ \text{"}X \text{ and } Y \text{ are homotopically equivalent"}\,. \end{cases}$

Let us now show, by counter-examples, that the converses of the above implications are not true in general.

Example 1. (π-equivalent $\not\Longrightarrow$ homeomorphic)

Let X be an indiscrete space containing more than one element, Y a topological space consisting of only one element. Since the T_0 partition space X^π of X is also a singleton which is of course homeomorphic to Y, X is π-equivalent to Y. But X and Y have different cardinalities; they can never be homeomorphic.

Example 2. (Lattice-equivalent $\not\Longrightarrow$ π-equivalent)

Let X be an infinite set with the co-finite topology. Then X is a T_1 space. Let $Y=X\cup\{z\}$ where z is an object not in X. Let the closed sets in Y be ϕ, Y and all finite subsets of X. Then Y is T_0 but not T_1. X and Y are clearly lattice-equivalent. But $X^\pi=X$ and $Y=Y^\pi$, hence they are not π-equivalent.

Example 3. (Homotopically equivalent $\not\Longrightarrow$ π-equivalent)

For T_1 spaces X and Y, X is π-equivalent to Y if and only if X is homeomorphic to Y. It is known that $[0, 1]$ and $(0, 1)$ are homotopically equivalent. But they are not π-equivalent.

3. π-invariant Properties.

Arising from the classification of topological spaces by π-equivalence, is the concept of π-invariant topological properties. We say that a topological property P is *π-invariant* if it is a property of all topological spaces of a π-equivalence class, i.e., if X and Y are π-equivalent and if X has the property P, then Y has the property P.

Clearly for any topological space X, X and X^π are π-equivalent; the following theorem shows that it is sufficient to test π-invariance for these typical pairs of π-equivalent spaces.

THEOREM 2.3. *Let P be a topological property. The following two statements are equivalent:*

(i) *P is a π-invariant property.*

(ii) *For any topological space X, X has P if and only if its T_0 partition space X^π has P.*

PROOF. (i) $\Longrightarrow$ (ii). Trivial.

(ii) $\Longrightarrow$ (i). Let X and Y be two π-equivalent topological spaces. If X has the property P, by (ii), X^π has also the property P. Since P is a topological property and X^π and Y^π are homeomorphic, Y^π has the property P. By (ii) again, Y has the property P.

In Theorems 2.1 and 2.2, we have proved that two π-equivalent spaces are lattice-equivalent as well as homotopically equivalent. The following theorem is then obvious:

THEOREM 2.4. *A topological property P is a π-invariant property if it is a lattice-invariant property or a homotopy property.*

It has been proved in [20] and [23] that the following topological properties are lattice-invariant:

Regularity, cmplete regularity, normality, compactness, local compactness[1], *being Lindelöf, second countability and connectedness.*

It has also been proved in [20] and [23] that the following topological properties are not lattice-invariant:

T_0, T_1, T_2, T_3, Tychonoff, complete normality, being separable and first countability.

Consequently we have

THEOREM 2.5. *Regularity, complete regularity, normality, compactness, local compactness, being Lindelöf, second countability and connectedness are π-invariant properties.*

The following example shows that none of the properties T_0, T_1, T_2 and Tychonoff is π-invariant.

Example. Let $I=[0,1]$ be the closed unit interval with the usual topology τ. Then I has the properties T_0, T_1, T_2, T_3 and Tychonoff. Let τ_1 be the family of τ-open sets that contain 1. Consider the one point extension $X=I\cup\{\infty\}$ with the topology $\tau^*=\{A\cup\{\infty\}: A\in\tau_1\}\cup(\tau\backslash\tau_1)$. In (X, τ^*), any neighbourhood of 1 contains ∞ and any neighbourhood of ∞ contains 1. Therefore, (X, τ^*) is a

[1] Here, by a locally compact space, we mean a topological space in which every point has a neighbourhood with compact closure.

non-T_0 space, and hence does not have any of the properties mentioned above. But I and X are clearly π-equivalent.

We shall show that separability, complete normality, first countability and pseudo-metrizability are all π-invariant properties. We state first a simple lemma:

LEMMA 2.6. *Let X be a topological space, X^π its T_0 partition space and p the natural projection. Then for any subsets A and B of X,*

(i) *$p(A^-)=p(A)^-$, and*

(ii) *$p(A\cap B)=p(A)\cap p(B)$ if one of A and B is open or closed.*

PROOF. (i) is an immediate consequence of the fact that p is a continuous and closed map. (ii) follows from the fact that for the open or closed set A, $A=p^{-1}(p(A))$.

THEOREM 2.7. *The following topological properties are π-invariant:*

(i) *Separability,*

(ii) *first countability,*

(iii) *complete normality, and*

(iv) *pseudo-metrizability.*

PROOF. Making use of Theorem 2.3, it is enough for us to show that X has a property P if and only if X^π has the property P.

(i) Suppose X is separable and A a countable dense subset of X. Then $p(A)^-=p(A^-)=p(X)=X$, by Lemma 2.6, i.e., $p(A)$ is a countable dense set in X^π. Hence X^π is separable. Conversely, let S be a countable dense subset in X^π. Consider the disjoint family $\{p^{-1}(s) : s\in S\}$ of non-empty sets in X. By Axiom of Choice, we have a subset B in X such that for each $s\in S$, $B\cap p^{-1}(s)$ is a singleton. B is obviously a countable subset of X. By Lemma 2.6, $p(B^-)=p(B)^-=S^-=X$. But B^- is closed. We have $B^-=p^{-1}(p(B^-))=p^{-1}(X^\pi)=X$. Therefore B is a countable dense subset in X.

(ii) By direct verification, one can easily see that

(a) if $U_1, \cdots, U_n, \cdots$ is a countable local base at x in X then $p(U_1), \cdots, p(U_n), \cdots$ is a countable local base at $p(x)$ in X; and

(b) if $W_1, \cdots, W_n, \cdots$ is a countable local base at $p(x)$ in X, then $p^{-1}(W_1), \cdots, p^{-1}(W_n), \cdots$ is a countable local base at x in X.

Clearly, from (a) and (b), (ii) follows.

(iii) It is known that a topological space X is completely normal if and only if, for any two sets A and B such that $A\cap B^-=\phi$ and $A^-\cap B=\phi$, there exist disjoint open sets U and V in X with $A\subset U$ and $B\subset V$ (See 11, p. 61). Thus it is enough to verify the following two statements:

(a) For any two sets A and B in X, $A\cap B^-=\phi$ if and only if $p(A)\cap p(B)^-=\phi$.

(b) For any two sets A and B in X, A and B have disjoint open neighbourhoods if and only if $p(A)$ and $p(B)$ have disjoint open neighbourhoods.

Proof of (a). It follows from Lemma 2.6 that

$$p(A)\cap p(B)^-=p(A)\cap p(B^-)=p(A\cap B^-)\ .$$

Therefore

$$p(A)\cap p(B)^-=\phi\ ,$$

if and only if $A\cap B^-=\phi$.

Proof of (b). If U, V are disjoint neighbourhoods of $p(A)$ and $p(B)$, then $p^{-1}(U)$ and $p^{-1}(V)$ are disjoint open neighbourhoods of A and B. It remains to show the converse. Let G and H be disjoint open neighbourhoods of A and B respectively. Then $p(G)$ and $p(H)$ are open neighbourhoods of A and B respectively. furthermore, by Lemma 2.6,

$$p(G)\cap p(H)=p(G\cap H)=p(\phi)=\phi\ .$$

(iv) It is well-known that a topological space X is pseudo-metrizable if and only if it possesses a σ-locally finite basis. Therefore, it suffices to show that for any topological space X, X possesses a σ-locally finite basis if and only if X^π does. Under the lattice isomorphism between the topologies of X and X^π (induced by the natural projection p), it is obvious that a family $\mathscr{B}$ of open sets in X forms a basis for X if and only if the corresponding family $\mathscr{B}^\pi$ forms a basis for X^π. So, what remains for us to show is that a family $\mathscr{A}$ of open sets is locally finite in X if and only if the corresponding family $\mathscr{A}^\pi$ is locally finite in X. This follows from Lemma 2.6 (ii) immediately.

CHAPTER III. t_1 Spaces and t_2 Spaces

In the last section of the previous chapter, we have tested the π-invariance of some well-known topological properties. Among those that are not π-invariant, we find the so-called separation axioms, $T_0, \cdots, T_4$. This is not at all too surprising, since we lump points together to construct T_0 partition spaces and the T_0 partition space usually satisfies stronger separation axiom than the original space. On the other hand, most topological properties are studied in conjunction with some separation axioms, therefore the theory of partition spaces and in particular the theory of π-invariance will be greatly enhanced if some sort of π-invariant separation axioms can be formed which are closely related to the well-known separation axioms. To find these, we take a separation axiom T_i, and look for a necessary and sufficient condition t_i that a topological space X satisfies so that the T_0 partition space X^π satisfies T_i. For T_0 no valuable result can be obtained since X^π always satisfies T_0. In the sequel we shall do this for T_1 and T_2 and treat T_3 and T_4 with t_1 and t_2.

1. The separation Axioms t_1 and t_2.

THEOREM 3.1. *Let X be a topological space and X^π its T_0 partition space. Then the following statements are equivalent.*

(i) *X^π is a T_1 space.*

(ii) *For all x, y in X if $x^- \cap y^- \neq \phi$, then $x^- = y^-$.*

PROOF. (i) $\Longrightarrow$ (ii). By Lemma 2.6, we have

$$p(x)^- \cap p(y)^- = p(x^-) \cap p(y^-) = p(x^- \cap y^-) .$$

If $x^- \cap y^- \neq \phi$, then it follows that $p(x)^- \cap p(y)^- \neq \phi$. Since X^π is a T_1 space, by assumption, we get $p(x) = p(y)$ which is the case if and only if x and y belong to the same closed sets of X. Therefore $x^- = y^-$.

(ii) $\Longrightarrow$ (i). Suppose $p(x) \neq p(y)$ in X^π. Then $x \notin y^-$ or $y \notin x^-$; therefore $x \notin y^-$ and $y \notin x^-$ by the assumption (ii). Consider the open sets $U = X \backslash y^-$ and $V = X \backslash x^-$ of X. Then $p(U)$ is an open set which contains $p(x)$. But $p(U) \cap p(y) = p(U \cap \{y\}) = p(\phi) = \phi$, therefore $p(U)$ is a neighbourhood of $p(x)$ in X^π which does not contain $p(y)$. Similarly $p(V)$ is a neighbourhood of $p(y)$ in X^π which does not contain $p(x)$. Therefore X^π is a T_1 space.

The above theorem leads us to define the separation axiom t_1 as the statement (ii) above. A topological space X is consequently called a t_1 *space* if it satisfies the conditions of Theorem 3.1. Clearly t_1 is a π-invariant topological property. Moreover if X is a T_1 space then X is a t_1 space since $X^\pi = X$; therefore t_1 is a weaker separation axiom than T_1.

THEOREM 3.2. *Let X be a topological space and X^π its T_0 partition space. Then the following statements are equivalent:*

(i) *X^π is a T_2 space.*

(ii) *For all x, y in X if $x^- \neq y^-$, then x and y have disjoint neighbourhoods.*

PROOF. (i) $\Longrightarrow$ (ii). Suppose X^π is a T_2 space and $x^- \neq y^-$. Then by Lemma 2.6, $x^- = p^{-1}(p(x^-)) = p^{-1}(p(x)^-) = p^{-1}(p(x))$ and similarly $y^- = p^{-1}(p(y))$. Therefore $p(x) \neq p(y)$. As X^π is a T_2 space, $p(x)$ and $p(y)$ have disjoint neighbourhoods in X^π. It follows from continuity of p that x and y have disjoint neighbourhoods in X.

(ii) $\Longrightarrow$ (i). Suppose $p(x) \neq p(y)$ in X. Then $x \notin y^-$ or $y \notin x^-$; hence $x^- \neq y^-$. By assumption, x and y have disjoint open neighbourhoods U and V respectively. On the other hand, $p(U \cap V) = p(U) \cap p(V)$, therefore $p(U)$ and $p(V)$ are disjoint open neighbourhoods of $p(x)$ and $p(y)$ respectively.

The above theorem leads us to define the separation axiom t_2 as the statement (ii) above. A topological space X is called a *t_2 space* if it satisfies the conditions of Theorem 3.2. Clearly t_2 is π-invariant topological property. Moreover if X is a T_2 space, then X is a t_2 space since $X^\pi = X$; therefore t_2 is a weaker separation axiom than T_2.

Remarks. The concept of t_2 space is implicit in a paper by *Dixmier* [3]. There a point x of a topological space is said to be *separated* if for each $y \notin x^-$, x and y have disjoint neighbourhoods. It is easily seen that a topological space X is a t_2 space if every point of X is separated.

The following theorem shows that, like their counterparts T_1 and T_2, the separation properties t_1 and t_2 are hereditary, productive but not divisible.

THEOREM 3.3.

(i) *Every subspace of a t_1 (respectively t_2) space is a t_1 (respectively t_2) space.*

(ii) *The product of a family of t_1 (respectively t_2) spaces is a t_1 (respectively t_2) space.*

(iii) *A quotient space of a t_1 (respectively t_2) space may not be a t_1 (respectively t_2) space.*

PROOF. (i) Suppose A is a subspace of a t_1 (respectively t_2) space X. Then A^π is homeomorphic to a subspace of X^π which is a T_1 (respectively T_2) space. Since T_1 (respectively T_2) is hereditary, A^π is a T_1 (respectively T_2) space. Therefore A is a t_1 (respectively t_2) space.

(ii) Let $(X_i)_{i \in I}$ be a family of spaces and P the topological product of X_i. Then the T_0 partition space P^π is homeomorphic to the topological product of the T_0 partition space X_i^π, i.e., $(\Pi X_i)^\pi \cong (\Pi X_i^\pi)$ (This follows from the fact that for every $x \in \Pi X_i$, $x^- = \Pi\{x_i\}^-$). Using similar argument as in the proof of (i), we easily prove (ii).

(iii) Let X be the set of all real numbers with usual topology. Then X is a t_1 and t_2 space. Consider the decomposition $X = \{0\} \cup \{x : x > 0\} \cup \{x : x < 0\}$. It is then easily seen that the quotient space of X corresponding to this decomposition is neither t_1 nor t_2.

2. Relationship among the Separation Axioms.

In the sequel we shall use the following abbreviations:

Ty: Tychonoff;
R : regularity;
CR: complete regularity; and
N : normality.

We know from general theory of topology that

$$\begin{array}{ccccccccccc} T_4 & \Longrightarrow & \mathrm{Ty} & \Longrightarrow & T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & & & & & \\ \mathrm{N} & & \mathrm{CR} & \Longrightarrow & \mathrm{R} & & & & & & \end{array}$$

and that the converses do not hold in general.

It follows from definitions that $t_2 \Longrightarrow t_1$. But the converse of this implication does not hold in general. For example if X is an infinite set with co-finite topology, then X is a t_1 space but not a t_2 space. It is easy to show that $R \Longrightarrow t_2$. In fact if X is a regular space and if $x^- \neq y^-$ in X, then either $y \notin x^-$ or $x \notin y^-$. It then follows from regularity of X that either y and x^- or x and y^- have disjoint neighbourhoods. Therefore X is a t_2 space. The diagram above can now be extended to

$$\begin{array}{ccccccccccc} T_4 & \Longrightarrow & \mathrm{Ty} & \Longrightarrow & T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \\ \mathrm{N} & & \mathrm{CR} & \Longrightarrow & \mathrm{R} & \Longrightarrow & t_2 & \Longrightarrow & t_1 & & \end{array}$$

We wish now to consider conditions under which the vertical arrows can be reversed. Recall that every T_0 space X is identical with its T_0 partition space X. Therefore it follows that X is a T_1 (respectively T_2) space if and only if X satisfies T_0 and t_1 (respectively t_2). We remark that t_1 does not imply T_1 nor does t_2 imply T_2 in general. To see this, it is sufficient to exhibit a t_2 space which is not T_1. Let $X=\{a, b, c\}$ and $\tau=\{\phi, X, \{a, b\}, \{c\}\}$ where a, b and c are distinct. Then X satisfies t_2 but not T_1.

It also follows from the above that a topological space X is a T_3 (respectively Tychonoff) space if and only if X is a T_0 space and a regular (respectively completely regular) space. If we recall that regularity and complete regularity are π-invariant properties, we see immediately that a necessary and sufficient condition for X^π to be T_3 (respectively Tychonoff) is that X is regular (respectively completely regular).

At this juncture it seems natural to extend the above diagram by defining a t_4 *space* to a normal space which satisfies t_1. We claim that $t_4 \Longrightarrow \mathrm{CR}$. Suppose F is a closed subset of a t_4 space X and $x \in X \backslash F$. Then $x^- \neq y^-$ for every

$y \in F$ since $x \notin y^- \subset F$. By the separation axiom t_1 we have $x^- \cap y^- = \phi$, and hence $x^- \cap F = \phi$. X being normal, there exists a real valued continuous function f which separates x^- and F, hence x and F. Therefore X is completely regular.

Finally for the sake of symmetry we may call any topological space a t_0 space. Summarizing, we have

$$\begin{array}{ccccccccccc} T_4 & \Longrightarrow & \mathrm{Ty} & \Longrightarrow & T_3 & \Longrightarrow & T_2 & \Longrightarrow & T_1 & \Longrightarrow & T_0 \\ \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow & & \Downarrow \\ t_4 & \Longrightarrow & \mathrm{CR} & \Longrightarrow & \mathrm{R} & \Longrightarrow & t_2 & \Longrightarrow & t_1 & \Longrightarrow & t_0 \end{array}$$

where the properties at the bottom row are all π-invariant properties and

$$\begin{array}{l} (T_0 \text{ and } t_0) \Longleftrightarrow T_0 \\ (T_0 \text{ and } t_1) \Longleftrightarrow T_1 \\ (T_0 \text{ and } t_2) \Longleftrightarrow T_2 \\ (T_0 \text{ and } \mathrm{R}) \Longleftrightarrow T_3 \\ (T_0 \text{ and } \mathrm{CR}) \Longleftrightarrow T_4 \\ (T_0 \text{ and } t_4) \Longleftrightarrow T_4 \,. \end{array}$$

We remark that although the definition of t_4 spaces may seem to be far fetched at first sight, there is quite a large class of important t_4 spaces. For example any regular space which satisfies the second axiom of countability or has a σ-locally finite basis is a t_4 space, so is also any pseudo-metrizable space. We shall see in the next section that any compact t_2 space is a t_4 space.

3. Applications.

We mentioned at the beginning of this chapter that since most topological properties are studied in conjunction with some separation axioms, it is most desirable to have some sort of π-invariant separation axioms for the study of π-invariant topological properties. In the previous section, we have found some of the π-invariant separation axioms, namely, t_0, t_1, t_2, R, CR and t_4, corresponding to which are the well-known topological separation axioms T_0, T_1, T_2, T_3, Ty and T_4 respectively. Suppose for example all T_1 spaces have a π-invariant property P. Then it would follow that all t_1 spaces have the same π-invariant property P. In other words any theorem about π-invariant properties of T_i spaces can be readily generalized into a theorem about the same π-invariant properties on the larger class of all t_i spaces. A slightly stronger principle of generalization is also true:

THEOREM 3.4. *Suppose P and Q are two π-invariant properties. If the im-*

plication $P \Longrightarrow Q$ is true for all T_i spaces ($i=0, 1, 2$), *then the implication is also true for all t_i spaces.*

PROOF. Let X be a t_i spaces which has the property P. Then X^π also has the property P, since P is π-invariant. By assumption the T_i space X^π has the property Q· Therefore X also has Q since Q is π-invariant.

COROLLARY. *If $P \Longrightarrow Q$ is true for all T_1 or all T_2 spaces, then $P \Longrightarrow Q$ is true for all regular spaces.*

In the sequel we shall make use of Theorem 3.4 and its corollary to obtain some non-trivial generalizations.

A. Compact, locally compact and supercompact spaces.

It is well-known that a compact Hausdorff space is a T_4 space. Since compactness and normality are both π-invariant properties, applying the principle above we obtain:

THEOREM 3.5. *A compact t_2 space is a t_4 space, and consequently it is normal and completely regular.*

In the literature one finds diverse definitions for locally compact space X such as

[LC1] Every $x \in X$ has a compact neighbourhood [11, p. 66].

[LC2] Every $x \in X$ has a neighbourhood with compact closure [12, p. 61].

[LC3] Every $x \in X$ has a closed compact neighbourhood [9].

[LC4] Every $x \in X$ has a compact local basis [3].

[LC5] Every $x \in X$ has a closed compact local basis [9].

Among these statements, [LC5] is the strongest and [LC1] is the weakest. For Hausdorff spaces it is known that [LC1] $\Longrightarrow$ [LC5] and hence all five statements are equivalent for Hausdorff spaces. We wish now apply the principle above to show

THEOREM 3.6. *Statements* [LC1], [LC2], [LC3], [LC4] *and* [LC5] *are all equivalent to each other for t_2 spaces.*

PROOF. It is enough to show [LC1] $\Longrightarrow$ [LC5] for t_2 spaces. By Theorem 3.4, it suffices to prove that [LC1] and [LC5] are π-invariant properties. But this follows from the lemma below:

LEMMA 3.7. *Let X/R be a partition space of a topological space X and p the natural projection. Then a subset K of X is compact if and only if $p(K)$ is compat.*

PROOF. It follows from continuity that if K is compact, then $p(K)$ is compact. Conversely if $p(K)$ is compact and $\mathscr{G}$ is an open cover of K in X, then $\{p(G): G\in\mathscr{G}\}$ is an open cover of $p(K)$ in X/R. Hence $p(K)$ has a finite subcover say $\{p(G_1), \cdots, p(G_n)\}$. It follows from $G_i=p^{-1}(p(G_i))$ that $\{G_1, \cdots, G_n\}$ is a finite cover of K.

In 1969, *J. de Groot* [7], defined supercompactness as follows:

A space is said to be supercompact if it possesses an open subbase $\mathscr{S}$ such that each subcollection of $\mathscr{S}$ covering the space contains a pair of sets which together cover the space.

A conjecture of de Groot that

Every compact metric space is supercompact

was proved by *J. L. O'Connor* in 1970 [14]. It is natural to ask whether we can have a similar theorem for pseudo-metric spaces, i.e., every compact pseudo-metric space is supercompact. A natural way is to go through the proof to see whether it still works; but an easy application of Theorem 3.4 saves all the tedious work. First of all the conjecture above can be rephrased as:

For any T_1 space X, if X is pseudo-metrizable and compact then X is supercompact.

It is known that pseudo-metrizability and compactness are π-invariant and it is easy to verify that supercompactness is also π-invariant. Making use of Theorem 3.4 and the fact that any pseudo-metrizable space is a t_1 space, we obtain the following theorem:

THEOREM 3.8. *Every compact pseudo-metrizable space is supercompact.*

B. Completely regular spaces.

Let X be a topological space. A base β for closed subsets of X is called a *normal base* if the following conditions are satisfied.

(i) Finite unions and finite intersections of members of β are still members of β.

(ii) For $B\in\beta$ and $x\in X\backslash B$ there is $A\in\beta$ such that $x\in A$ and $A\cap B=\phi$.

(iii) For disjoint A and B in β there are $C, D\in\beta$ such that $X\backslash C$ and $X\backslash D$ are disjoint neighbourhoods of A and B.

A topological space is called *semi-normal* if it possesses a normal base. *O. Frink* [5] proves that a T_1 space is completely regular if and only if it is semi-normal. Since every completely regular space is t_1 and complete regularity is π-invariant.

THEOREM 3.9. *A topological space is completely regular if and only if it is semi-normal.*

will follow from

LEMMA 3.10. *Every semi-normal space is a t_1 space and semi-normality is a π-invariant property.*

PROOF of Lemma. Let X be a semi-normal space and let β be a normal base of X. Suppose $x^- \neq y^-$ in X. Then, either $x \notin y^-$ or $y \notin x^-$. For the former case $x \notin y^-$, we can find $B \in \beta$ such that $y^- \subset B$ and $x \in X \backslash B$ since β is a base for closed subsets in X. Using condition (ii) above we have $A \in \beta$ such that $x \in A$ and $A \cap B = \phi$. By condition (iii) we obtain disjoint neighbourhoods for A and B which are disjoint neighbourhoods of x and y. Therefore X is a t_2 and hence a t_1 space.

For the second part of the lemma, it is enough to show that a topological space Y is semi-normal if and only if Y^π is semi-normal. If β is a normal base of Y, then it is easy to verify directly that $\beta^* = \{p(B) : B \in \beta\}$ is a normal base of Y^π. Conversely for any normal base γ of Y^π, $\bar{\gamma} = \{p^{-1}(C) : C \in \gamma\}$

Remarks. We have shown in Theorem 3.9 that the T_1 condition from *O. Frink's* theorem can be removed. *J. de Groot* and *J. M. Aarts* [8] show that the semi-ring condition (i) above can be removed while retaining the T_1 condition. *E. F. Steiner* [17] proves that a topological space is completely regular if and only if it possesses a normal separating family of closed sets. *E. F. Steiner* doubts of the possibility of modifying *O. Frink's* proof to cover all completely regular spaces as what we have done here.

CHAPTER IV. COMPACTIFICATIONS

The compactifications yield useful results if a Hausdorff space X to be compactified and its compactifications are Hausdorff spaces. In the sequel, we shall relax this condition and consider compactifications of t_2 spaces.

1. Aleksandrov Compactification.

Let X be a topological space and $\hat{X} = X \cup \{\infty\}$ the Aleksandrov one point compactification of X. It is well-known that

(a) $\hat{X}$ is a compact space,

(b) X is an open subspace of the space $\hat{X}$,

(c) if X is a non-compact space, then X is dense in $\hat{X}$, and

(d) $\hat{X}$ is a Hausdorff space if and only if X is a locally compact Hausdorff space.

We shall show that for t_2 spaces, a statement similar to (d) above holds.

THEOREM 4.1. *A necessary and sufficient condition of the Aleksandrov one point compactification $\hat{X}$ of a topological space X being t_2 is that X is a locally compact t_2 space.*

PROOF. *Necessity*: Since any subspace of a t_2 space is also t_2, X is a t_2 space. X is open in $\hat{X}$ which is a t_2 space, therefore, for every $x \in X$, x and ∞ have disjoint open neighbourhoods, U and V respectively. Now $X \backslash V$ contains U and is compact, therefore, it is a compact neighbourhood of x in X.

Sufficiency: For any $x \in X$, let C be a compact neighbourhood of x in X. By Theorem 3.6, we may assume that C is a closed subset of X. Thus, C and $\hat{X} \backslash C$ are disjoint neighbourhoods of x and ∞ respectively. In other words, any $x \in X$ is strongly separated from ∞ in $\hat{X}$. Suppose $y \in X$ has a neighbourhood U in $\hat{X}$ which does not contain x. Then either U or $U \backslash \{\infty\}$ is a neighbourhood in X of y which does not contain x. Since X is t_2, x and y have disjoint neighbourhoods in X. But these are also disjoint neighbourhoods of x and y in $\hat{X}$. Therefore, $\hat{X}$ is a t_2 space.

2. Stone-Čech Compactifications of Completely Regular Spaces.

Let X be a topological space. A *compactification* of X is defined to be a pair (f, Y) where Y is a compact topological space and f is a homeomorphism of X onto a dense subspace of Y. A compactification (f, Y) is called Hausdorff (respectively t_2) if and only if Y is a Hausdorff (respectively t_2) space. It is well-known that a topological space X admits a Hausdorff compactification if and only if it is a Tychonoff space. We show first that only completely regular spaces can have a t_2-compactification.

LEMMA 4.2. *If a topological space X has a t_2-compactification, then X is a completely regular space.*

PROOF. Let (f, Y) be a t_2-compactification of X. Then Y is a compact t_2-space and hence a completely regular space by Theorem 3.5. As complete regularity is a hereditary property, $f(X)$ and hence also X are completely regular spaces.

Our next problem is to show that every completely regular space admits a t_2-compactification which has similar features as those of the *Stone-Čech* compactifications of Tychonoff spaces. It turns out that in relation to the separation axioms, this compactification falls between t_2 and T_2. This leads us to the following definition.

We say that a t_2-compactification (f, Y) of a topological space X is a T_2/t_2-

compactification of X if every point of $Y \backslash f(X)$ is a *Hausdorff point* of Y, i.e., if every pair of points t of $Y \backslash f(X)$ and y of Y have disjoint neighbourhoods in Y. We now show that every completely regular space has a T_2/t_2-compactification. We recall that in the construction of the Stone-Čece compactification of a Tychonoff space X, the assumption that X is a Hausdorff space plays an important role in assuring that the family F of continuous functions of X into $Q=[0, 1]$ distinguishes points. This method no longer applies to the present case of compactifying t_2 spaces; but the forming of T_0 partition spaces provides us a convenient devise to overcome the difficulty.

THEOREM 4.3. *Let X be a completely regular space; let X^π be the T_0 partition space of X (note that X^π is now a Tychonoff space) and $p: X \to X^\pi$ the natural projection. If (h, Y) is a Hausdorff compactification of the Tychonoff space X^π, then there exists a T_2/t_2-compactification (k, Z) of X such that Y is the T_0 partition space of Z and the diagram*

$$\begin{array}{ccc} X & \xrightarrow{k} & Z \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X^\pi & \xrightarrow{h} & Y = Z^\pi \end{array}$$

is commutative where $q: Z \to Z^\pi$ is the natural projection.

PROOF. Let Z be the disjoint union of the set X and the set $Y \backslash h(X^\pi)$ and define $q: Z \to Y$ by putting

$$q(t) = \begin{cases} h(p(t)) & \text{if } t \in X \\ t & \text{if } t \in Y \backslash h(X^\pi) . \end{cases}$$

Then q is a surjective mapping of Z onto Y. Let Z be given the coarsest topology so that q is continuous, i.e., a subset G of Z is open if and only if $G=q^{-1}(U)$ for some open set U of Y. Then Y is the T_0 partition space of Z. i.e., $Y=Z^\pi$, and $q: Z \to Y$ is the natural projection. Moreover, it follows that Z is a compact t_2 space. We define now $k: X \to Z$ to be the inclusion mapping. Then the diagram of the theorem is commutative, and it follows from Lemma 1.9 that k is continuous. We shall now prove that k is an imbedding. Let G be an open set of X. Since p is open and h is an imbedding, $q(k(G))= h(p(G))$ is open in $h(X^\pi)$. Let U be an open set of Y so that $U \cap h(p(X))=q(k(G))$. By the commutativity of the diagram, we have $U \subset q(k(X))=q(k(G))$. It follows from the definition of q and k and the assumption that G is open in X that $k(G)=G=q^{-1}(q(G))=q^{-1}(q(k(G)))=q^{-1}(U \cap q(k(X)))=q^{-1}(U) \cap k(X)$. But $q^{-1}(U)$ is open in Z, therefore $k(G)$ is open in $k(X)$, and hence k is an imbedding. We are now going to prove that $k(X)$ is dense in Z. Let U be a non-empty open set of Z. Then $q(U)$ is a non-empty open set in Y. Since $h(p(X))$ is dense in Y, there is $x \in X$ such that $h(p(x)) \in q(U)$. It follows from the commutativity of the diagram that $q(k(x)) \in q(U)$. Since Y is the T_0 partition of Z and q is the natural projection, we have $U = q^{-1}(q(U))$. Therefore $k(x) \in q^{-1}(q(k(x))) \subset U$ proving that $k(X)$ is dense in Z. Thus, (k, Z) is a t_2-compactification of X. It remains now to prove that the points of $Z \backslash k(X)$ are Hausdorff in Z. For any $t \in Z \backslash k(X)$ and any $z \in Z$ distinct from t, we have $q(t) \neq q(z)$ by the definition of q. Since Y is Hausdorff, $q(t)$ and $q(z)$ have in Y disjoint neighbourhoods which give rise to disjoint neighbourhoods of t and z in Z. Therefore (k, Z) is a T_2/t_2-compactification of X. The proof is now complete.

Let us now look at some consequences of Theorem 4.3. Since every Tychonoff space has a Hausdorff compactification, Theorem 4.3 together with Lemma 4.2 yield a characterization of completely regular topological spaces:

COROLLARY. *A topological space X is homeomorphic to a subspace of a compact t_2 space if and only if X is a completely regular space.*

If, in the proof of Theorem 4.3, we make use of the Stone-Čech compactification $(e, \beta(X^\pi))$ of the Tychonoff space X^π instead of an arbitrary Hausdorff compactification (h, Y), then we obtain a unique T_2/t_2-compactification $(f, \gamma(X))$ of the completely regular space X, such that the diagram

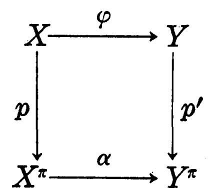
$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & \gamma(X) \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle q} \\ X^\pi & \xrightarrow{\ e\ } & \beta(X^\pi) = \gamma(X)^\pi \end{array}$$

is commutative. Suppose the completely regular space X is itself a Tychonoff space, then $X^\pi = X$ and the construction yields $(f, \gamma(X)) = (e, \beta(X))$. Therefore it is justified to define the T_2/t_2-compactification $(f, \gamma(X))$ of the completely regular space X as the *Stone-Čech compactification of the completely regular space X.* Clearly this compactification has the property that $f(X)$ is dense in $\gamma(X)$; let us now establish the other characterizing property of a Stone-Čech compactification which permits continuous extension of continuous functions. More precisely, we prove:

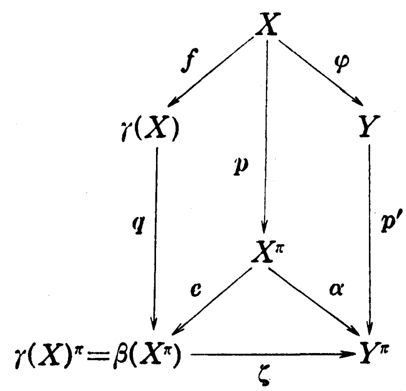
THEOREM 4.4. *Let X be a completetely regular space and $(f, \gamma(X))$ the Stone-Čech compactification of X. If φ is a continuous function on X to a com-*

pact t_2 *space* Y, *then there exists a continuous function* ψ *on* X^π *to* Y *such that* $\psi\circ f=\varphi$.

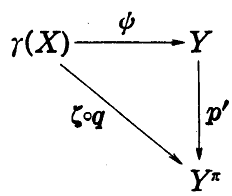
PROOF. Consider the T_0 partition spaces X^π and Y^π of X and Y. By Theorem 1.11, a unique continuous function $\alpha: X^\pi \to Y^\pi$ exists such that the diagram



is commutative where p and p' are the natural projections. Now α is a continuous function of the Tychonoff space X^π into the compact Hausdorff space Y^π, therefore, by the well-known property of Stone-Čech compactification of Tychonoff spaces, there exists a continuous function $\zeta: \beta(X^\pi) \to Y^\pi$ such that $\alpha=\zeta\circ e$. Putting all these together we obtain a commutative diagram as follows:



Define now a function $\psi: \gamma(x) \to Y$ by putting $\psi(x)=\varphi(x)$ if $x\in X$ and $\psi(x)=y$ such that $p'(y)=\zeta\circ q(x)$ if $x\notin X$. Clearly $\psi\circ f=\varphi$; it remains to prove that ψ is continuous. By Lemma 1.9, it is sufficient to show that the following diagram is commutative:



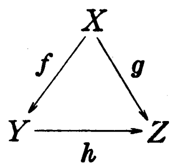
For $x\in X$, we get

$\zeta\circ q(x)=\zeta\circ q\circ f(x)=\zeta\circ e\circ p(x)=\alpha\circ p(x)=p'\circ\varphi(x)=p'\circ\psi(x)$. On the other hand, if $x\in\gamma(X)\backslash X$, then $p'\circ\psi(x)=\zeta\circ q(x)$ by definition of ψ. Therefore $\zeta\circ q=p'\circ\psi$ and the proof is complete.

The above theorem can be rephrased as: every continuous function φ on a completely regular space X into a compact t_2 space has a continuous extension over the Stone-Čech compactification $\gamma(X)$.

3. Characterization of $\gamma(X)$.

A relation is defined on the collection of all compactifications of a topological space X by agreeing that $(f, Y)\geqslant(g, Z)$ if and only if there is a continuous map h of Y into Z such that $h\circ f=g$:



Equivalently $(f, Y)\geqslant(g, Z)$ if and only if the function $g\circ f^{-1}$ on $f(X)$ to Z has a continuous extension h which carries Y into Z. If the function h can be taken to be a homeomorphism, then (f, Y) and (g, Z) are said to be *topologically equivalent*. In this case both of the relations $(f, Y)\geqslant(g, Z)$ and $(g, Z)\geqslant(f, Y)$ hold. It is well-known that (a) the collection of all compactifica tions of a topological space is partially ordered by $\geqslant$ which is not necessarily anti-symmetric and (b) if (f, Y) and (g, Z) are Hausdorff compactifications of a space and $(f, Y)\geqslant(g, Z)\geqslant(f, Y)$, then (f, Y) and (g, Z) are topologically equivalent. Consequently, the Stone-Čech compactificatioP $(e, \beta(X))$ of a Tychonoff space $\beta(X)$ is characterized by the extension property that evyry continuous function f on X to a compact Hausdorff space can be extended continuously to $\beta(X)$. We wish to establish an analogous result that the Stone-Čech compactification $(f, \gamma(X))$ of a completely regular space is characterized by the extension property of Theorem 4.4.

In proving (b) mentioned above, one makes use of the fact that, for any Hausdorff space X, the identity map of X is the only continuous extension of the inclusion map of a dense subspace A into X. We now prove an analogous lemma.

LEMMA 4.5. *Let X be a topological space and A a dense subspace of X such that the points of $X\backslash A$ are Hausdorff in X. The identity map of X is the only continuous extension of the inclusion map of A into X.*

PROOF. Let $j : A \to X$ be the inclusion map and let f be a continuous extension of j over X. We are going to show that f is the identity map of X. Suppose there exists $x \in X$ such that $f(x) \neq x$. Then x cannot belong to A and hence it is a Hausdorff point in X. x and $f(x)$ therefore have disjoint neighbourhoods. Since A is dense in X, there is a net N in A such that $N \to x$. By continuity of f and $f|_A = i_A$, we have $N = f \circ N \to f(x)$, contradicting the fact that x and $f(x)$ have disjoint neighbourhoods.

The T_2/t_2-version of property (b) can be proved.

COROLLARY 4.6. *If (f, Y) and (g, Z) are T_2/t_2-compactification of a topological space X and $(f, Y) \geqslant (g, Z) \geqslant (f, Y)$, then (f, Y) and (g, Z) are topologically equivalent.*

PROOF. If (f, Y) and (g, Z) and T_2/t_2-compactifications of X each of which follows the other relative to the ordering $\geqslant$, then both $f \circ g^{-1}$ [1)] and $g \circ f^{-1}$ have continuous extensions j and k to all of Z and Y respectively. Since $k \circ j$ is the identity map on the dense subset $g(X)$ of Z and Z is a compact t_2 space such that points of $Z \backslash g(X)$ are Hausdorff, $k \circ j$ is the identity map of Z onto itself. Similarly $j \circ k$ is the identity map of Y onto Y. Consequently (f, Y) and (g, Z) are topologically equivalent.

It follows now that (i) any T_2/t_2-compactification (g, Z) of a completely regular space X is topologically equivalent to the Stone-Čech compactification $(f, \gamma(X))$ if every continuous function of X into a compact t_2 space Y can be extended continuously to Z and (ii) the Stone-Čech compactification $(f, \gamma(X))$ is a maximal T_2/t_2-compactification of the completely regular space X.

4. t_2-compactifications.

Though the results of the previous section may not hold for the less restrictive collection of all t_2-compactifications of a completely regular space, some interesting results are available. We first show that the construction of t_2-compactifications is compatible with the forming of T_0 partition spaces.

LEMMA 4.7. *Let X be a completely regular space, and (f, Y) a t_2-compactification of X. Then (f^π, Y^π) is a Hausdorff compactification of the Tychonoff space X^π.*

PROOF. Clearly we have a commutative diagram

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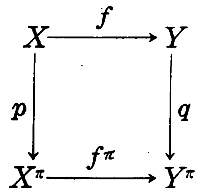
1) Here and in the similar cases in the sequel, g is regarded as a homeomorphism from X onto $g(X)$. So g^{-1} is defined on the subspace $g(X)$ of Y.

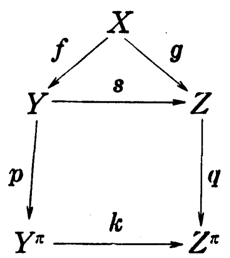


where Y^π is a compact Hausdorff space. Moreover, it follows from Theorem 1.12 that f^π is an imbedding. It remains to prove that $f^\pi(X^\pi)$ is dense in Y^π. Now let U be a non-empty open set in Y^π. Then $q^{-1}(U)$ is a non-empty open set of Y. Since $f(X)$ is dense, there is an element x of X such that $f(x) \in q^{-1}(U)$. Now $f^\pi(p(x))=q(f(x)) \in U$ therefore $f^\pi(X^\pi)$ is dense in Y, and the theorem is proved.

Next we prove that the forming of T_0 partition is an increasing mapping relative to the partial ordering $\geqslant$.

LEMMA 4.8. *Let X be a completely regular space and let (f, Y) and (g, Z) be two t_2-compactifications of X. Then $(f, Y) \geqslant (g, Z)$ if and only if $(f^\pi, Y^\pi) \geqslant (g^\pi, Z^\pi)$.*

PROOF. Suppose $(f, Y) \geqslant (g, Z)$ and h is a continuous map of Y into Z such that $h \circ f = g$. Then it follows from the functorial property of π, that h^π is a continuous map of Y^π into Z^π such that $h^\pi \circ f^\pi = (h \circ f)^\pi = g^\pi$. Therefore $(f^\pi, Y^\pi) \geqslant (g^\pi, Z^\pi)$. Conversely, suppose $(f^\pi, Y^\pi) \geqslant (g^\pi, Z^\pi)$ and k is a continuous map of Y^π onto Z^π such that $k \circ f^\pi = g^\pi$. Denoting by $p: Y \to Y^\pi$ and $q: Z \to Z^\pi$ the natural projections, we define a map s of Y into Z by putting $s(f(x))=g(x)$ for every $x \in X$, and $s(y)=z$ where $q(z)=k(p(y))$ for $y \in Y \backslash f(X)$. Then the diagram



is commutative and hence s is continuous by Lemma 1.9. Therefore $(f, Y) \geqslant (g, Z)$.

COROLLARY 4.9. *If (f, Y) and (g, Z) are t_2-compactifications of a topological space X and $(f, Y) \geqslant (g, Z) \geqslant (f, Y)$, then there is a π-equivalence s of Y into Z (i.e. $s: Y \to Z$ is continuous and s^π is a homeomorphism of Y^π onto Z^π) such that $s \circ f = g$.*

PROOF. Notations are as in the proof of Lemma 4.8. It follows from Lemma 4.8 that $(f^\pi, Y^\pi) \geqslant (g^\pi, Z^\pi) \geqslant (f^\pi, Y^\pi)$. Since these are Hausdorff compactifications of X^π, we have a homeomorphism $k: Y^\pi \to Z^\pi$ such that $k \circ f^\pi = g^\pi$. Lifting k to $s: Y \to Z$ as in the proof of Lemma 4.8, we obtain a π-equivalence s such that $s \circ f = g$.

5. Wallman-Frink Compactification.

Let X be a T_1 space. Let $\mathscr{F}$ be the family of all closed subsets of X, and let $W_{\mathscr{F}}(X)$ be the collection of all subfamilies $\mathscr{A}$ of $\mathscr{F}$ which possess the finite intersection property and are maximal in $\mathscr{F}$ relative to this property. For each open subset U of X, let $U^* = \{\mathscr{A} : \mathscr{A} \in W_{\mathscr{F}}(X)$ and $A \subset U$ for some A in $\mathscr{A}\}$. Give to the set $W_{\mathscr{F}}(X)$ the topology with a base the family of all sets of the form U^* for U open in X; then the topolgical space $W_{\mathscr{F}}(X)$ so obtained is compact. Define a map $\varphi : X \to W_{\mathscr{F}}(X)$ by putting $\varphi(x) = \{A : A \in \mathscr{A}$ and $x \in A\}$; then φ is an imbedding, and $\varphi(X)$ is dense in $W_{\mathscr{F}}(X)$. The pair $(\varphi, W_{\mathscr{F}}(X))$ is called the *Wallman* compactification of X. It is known that

(a) if X is a T_4 space, then $(\varphi, W_{\mathscr{F}}(X))$ is a Hausdorff compactification; and in this case, it is topologically equivalent to the Stone-Čech compactification, and

(b) if f is a bounded continuous real-valued function on X, then $f \circ \varphi^{-1}$ may be extended continuously to all of $W_{\mathscr{F}}(X)$.

O. Frink [5] proved in 1964 the following theorem:

A T_1 space X is a completely regular space if and only if it possesses a normal base $\mathscr{B}$ for closed sets (see Chapter III, § 3B).

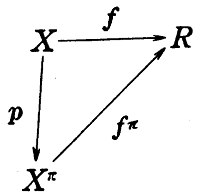
Using a normal base $\mathscr{B}$ for closed sets of a Tychonoff space X instead of $\mathscr{F}$ in the above construction, he was able to obtain a Hausdorff compactification $(\varphi, W_{\mathscr{B}}(X))$. This Wallman-Frink compactification has the property (c): for any continuous real-valued function f on X, $f \circ \varphi^{-1}$ can be extended continuously to all of $W_{\mathscr{B}}(X)$ if and only if f is $\mathscr{B}$-uniformly continuous (see definition below).

In the sequel, we shall drop the separation axiom T_1 and establish similar results for completely regular spaces.

Let X be a completely regular space and $\mathscr{B}$ a normal base for closed sets of X. (By Theorem 3.9, $\mathscr{B}$ exists). A real-valued function f defined on X is said to be *$\mathscr{B}$-uniformly continuous* if for any given $\delta > 0$ there exists a finite open cover of X by $\mathscr{B}$-complements (i.e., complements of members of $\mathscr{B}$) such that for any x, y in one and the same member of this cover $|f(x) - f(y)| < \delta$.

THEOREM 4.12. *Let X be a completely regular space and $\mathscr{B}$ a normal base for closed sets of X. Let $\mathscr{C} = \{p(A) : A \in \mathscr{B}\}$ be the corresponding normal base for closed sets of X^π. A real-valued function f defined on X is $\mathscr{B}$-uniformly* *continuous if and only if f^{π} is $\mathscr{C}$-uniformly continuous.*

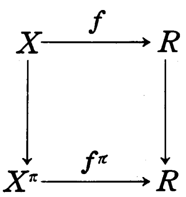
PROOF. Since the real line R is a T_2 space, we have $R^{\pi}=R$ and the diagram



commutes. If f is $\mathscr{B}$-uniformly continuous, then for given $\delta>0$, there exists a finite open cover $\{G_1, \cdots, G_n\}$ of $\mathscr{B}$-complements such that the oscillation of f on each G_i is less than δ. By the construction of $\mathscr{C}$, and the surjectivity of p, the family $\{p(G_1), \cdots, p(G_n)\}$ is a finite open cover of X, and $p(G_i)$ is $\mathscr{C}$-complement for each i. For y_1, y_2 in the same $p(G_i)$, let $x_1, x_2 \in X$ such that $p(x_1)=y_1$ and $p(x_2)=y_2$. Then $|f^{\pi}(y_1)-f^{\pi}(y_2)|=|f(x_1)-f(x_2)|<\delta$. This proves that f^{π} is $\mathscr{C}$-uniformly continuous on X. The converse part can be proved similarly.

THEOREM 4.13. *Let X be a completely regular space. A real-valued function f defined on X can be extended continuously over a t_2-compactification Y of X if and only if f^{π} can be extended continuously over Y^{π}.*

PROOF. Consider the following commutative diagram:



where R is the real line. The theorem follows from Theorem 1.15 immediately.

O. Frink's theorem [5], on extension of a real-valued function is as follows:

A real-valued function $f(x)$ defined over a semi-normal space x with normal space X with normal base Z can be extended to a real continuous function over the compactification $W(Z)$ if and only if $f(x)$ is Z-uniformly continuous.

In the above statement, a semi-normal space means a T_1 space which has a normal base. Therefore, equivalently, X is a Tychonoff space. It follows from Theorem 4.12 and Theorem 4.13 that we can relax the T_1 condition from O. Frink's theorem. Actually, as a consequence of Theorem 4.12, Theorem 4.13 and O. Frink's theorem mentioned above, we get the following theorem:

THEOREM 4.14. *Let X be a completely regular space and $\mathscr{B}$ a normal base in X. A real-valued function defined on X can be extended to a continuous real-*

valued function over the Wallman-Frink compactification $W_{\mathscr{B}}(X)$ if and only if f is $\mathscr{B}$-uniformly continuous on X.

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