A COUNTEREXAMPLE TO A GENERALIZED LOOP THEOREM

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In 1960 *J. Stallings* proved a combined form of *Dehn*'s lemma and the loop theorem [2]. He also gave a counterexample to the following.

Conjecture: *If M is a compact 3-manifold and F a closed (tame) surface embedded in* $\operatorname{int} M$, *such that* $\ker(i_*\colon \pi_1(F)\to\pi_1(M))\neq 1$ (*where* $i\colon F\to M$ *denotes inclusion*), *then there exists a simple closed curve J on F such that $J\not\simeq 1$ on F, and J bounds a non-singular disc D in M, with $D\cap F=\partial D=J$.*

If F is two-sided in M, this conjecture is true [1]. In *Stalling*'s counterexample F is a non-orientable closed surface in an orientable 3-manifold.

In 1969 *Suzuki* proved the following theorem:

If $F_{p,q}$ is an orientable surface of genus p with q boundary components embedded in $\operatorname{int}(M)$, such that $i_*\colon \pi_1(\operatorname{int} F_{p,q})\to\pi_1(M-\partial F_{p,q})$ is not an isomorphism (into), then there exists a simple closed curve J on $F_{p,q}$, $J\not\simeq 1$ on $F_{p,q}$, and J bounds a non-singular disc D with $\operatorname{int} D\subset M^3-F_{p,q}$.

We remark that this theorem is true if $F_{p,q}$ is 2-sided embedded in M, ($F_{p,q}$ orientable or not), and give two counterexamples to the theorem if M is non-orientable.

1) $q=0$, $p=1$.

Let $M=P^2\times S^1$ (P^2 is the projective plane). Let $\pi_1(M)=\boldsymbol{Z}_2(\alpha)\times\boldsymbol{Z}(\beta)$, where α is represented by a simple loop in P^2 and β by a simple loop in S^1. Let $F_{1,0}=\alpha\times\beta$, a (one-sided) torus in M. Clearly $i_*\colon \pi_1(F_{1,0})\to\pi_1(M)$ is not an isomorphism. If J is any closed curve on $F_{1,0}$ then $J\simeq m\alpha+n\beta$, $(m,n)=1$.

If J bounds a disc in M, then $n=0$, $m\equiv 0 \pmod 2$. But if $m\neq 0$ then J can not be represented by a simple closed curve. Thus there exists no simple closed curve as in the theorem.

2) $q=2$, $p=0$.

Again, let $M=P^2\times S^1$. Let k be a simple arc in S^1. Let $F_{0,2}=\alpha\times k$, a (one-sided) annulus in M. Then $i_*\colon\pi_1(\operatorname{int} F_{0,2})\to\pi_1(M-\partial F_{0,2})$ is not an isomorphism, for if t is a simple closed curve on $\operatorname{int} F_{0,2}$ that generates $\pi_1(\operatorname{int} F_{0,2})$, we have that $i(t)$ lies on a projective plane in $M-\partial F_{0,2}$ and therefore $i_*(t^2)=1$.

If J is any simple closed curve on $F_{0,2}$ that is not $\simeq 1$ on $F_{0,2}$, we have $J \simeq \alpha$. But such a J can not bound any disc in M.

Remark. The second counterexample can clearly be modified by taking $M=P^2 \times I$ and letting $F_{0,2}=\alpha \times I$.

REFERENCES

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