ON RANDOM TRANSLATIONS OF POINT PROCESSES

By

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1. Introduction. The purpose of this paper it to study point processes and their random translations by a method which is a little different from those already published (say *Dobrushin* [5], *Goldman* [8], and *Stone* [13]). It seems that most of the published results on random translations of point processes are those on the space R^d or its subgroups. In this paper we consider point processes on an arbitrary locally compact separable metric space X.

In Section 2 we introduce a metric into the class S of all measures in the space X, and define point processes and random measures as probability measures in this metric space S. If $X=R^d$, then this definition coincides essentially with the usual definition by *Ryll-Nardzewski* [12]. The convergence of random measures and point processes is defined as the weak convergence of probability measure on S. In particular, if $X=R^d$ and the limiting point process is Poisson, then our definition of convergence coincides with a definition in [5] and [13]. In Section 3, we prove two existence theorems for random measures and point processes. Although theorems in these two sections are stated in terms of generating functions and functionals, it seems that they are essentially included in known results (see [9] and [10]). In Section 4 we define mixed Poisson process generated by a random measure. These point processes are generalizations of those considered in [5] and others. In Section 5 is treated random translations of point processes, and it is shown that the limit of the random translations of any point process, if exists, must be mixed Poisson. The convergence to a Poisson process was treated in [5], [8] and [13], with some restrictions on the initial point processes. We give a condition for the convergence to a Poisson process without additional assumptions on the initial processes. In Section 6, we study the structure of point processes which are invariant under a given random translation, and prove a theorem which is essentially a strengthening of results of [5] and [8].

2. Random measures and point processes. Let X be a locally compact separable metric space with a metric d. Let $C_0(X)$ denote the class of all continuous real functions φ on X such that $\operatorname{supp}[\varphi]=\overline{\{x;\varphi(x)\neq 0\}}$ is compact. Let $C_0^+(X)$ denote the subset of $C_0(X)$ consisting of all non-negative $\varphi\in C_0(X)$.

For φ_n, $n\geq 1$, and φ in $C_0(X)$, $\lim_{n\to\infty}\varphi_n=\varphi$ means that the following two conditions are satisfied: (i) there exists a compact set $K\subset X$ such that $\operatorname{supp}[\varphi_n]\subset K$, $n\geq 1$, and (ii) φ_n converges to φ uniformly on K. We can use Stone-Weierstrass theorem to show the existence of the sequence $\{\alpha_n\}$, $\alpha_n\in C_0^+(X)$ having the following property: for every $\varphi\in C_0(X)$ there exists a sequence $\{\varphi_n\}$, where φ_n is a finite linear combination of α_k, such that $\lim_{n\to\infty}\varphi_n=\varphi$.

Let $\mathfrak{B}$ be the class of all Borel subsets of X, S the class of all measures on $\mathfrak{B}$, where we mean by a measure a non-negative countably additive set function μ on $\mathfrak{B}$ such that $\mu(A)<\infty$ if A is compact. For $\mu\in S$ and a real measurable function φ on X, we write $(\varphi,\mu)=\int_X\varphi d\mu$ if the integral on the right exists.

Let R^∞ denote the space of all sequences $\omega=(\omega_1,\omega_2,\cdots)$ of real numbers. $\rho_0(\omega,\omega')=\sum_{k=1}^{\infty}2^{-k}|\omega_k-\omega'_k|(1+|\omega_k-\omega'_k|)^{-1}$ defines a metric in R^∞. With this metric R^∞ is a complete separable metric space (see [1] p. 218). The classes of Borel sets of R^d and R^∞ are denoted by $\mathfrak{R}^d$ and $\mathfrak{R}^\infty$ respectively.

Let τ be a mapping from S to R^∞ defined by $\tau(\mu)=(\omega_1,\omega_2,\cdots)$, $\mu\in S$, $\omega_i=(\alpha_i,\mu)$. Since $(\alpha_i,\mu)=(\alpha_i,\nu)$, $i\geq 1$, implies that $\mu=\nu$, τ is one-to-one. Hence $\rho(\mu,\nu)=\rho_0(\tau(\mu),\tau(\nu))$ defines a metric in S. We write $\mu_n\xrightarrow{w}\mu$ if $\rho(\mu_n,\mu)\to 0$. Note that $\mu_n\xrightarrow{w}\mu$ if and only if $(\varphi,\mu_n)\to(\varphi,\mu)$ for every $\varphi\in C_0(X)$.

Lemma 2.1. *$\tau(S)$ is closed in R^∞, and (S,ρ) is a complete separable metric space.*

Proof. If $\{\mu_n\}$, $\mu_n\in S$, is a Cauchy sequence, then so is $\{(\varphi,\mu_n)\}$ for $\varphi\in C_0(X)$. $I(\varphi)=\lim_{n\to\infty}(\varphi,\mu_n)$ defines a positive linear functional I on $C_0(X)$. Hence there exists a $\mu\in S$ such that $I(\varphi)=(\varphi,\mu)$, $\varphi\in C_0(X)$, and $\mu_n\xrightarrow{w}\mu$. This proves that S in complete and $\tau(S)$ is closed. S is separable since S is homeomorphic to $\tau(S)\subset R^\infty$.

Let S_A denote the class of all measures μ such that $\mu(A^c)=0$, where A is a closed subset of S. Let S^p denote the class of all measures $\mu\in S$ such that $\mu(A)$ is a non-negative integer for every bounded set $A\in\mathfrak{B}$, where by a bounded set we mean a set $A\subset X$ such that $\bar{A}$ is compact.

Lemma 2.2. *S_A and S^p are closed in S.*

Proof. It is easy to see that S_A is closed. To prove that S^p is closed we suppose $\mu_n \in S^p$, $\mu_n \xrightarrow{w} \mu \in S$. If $A \in \mathfrak{B}$ is bounded and if $\mu(\partial A)=0$, where ∂A is the boundary of A, then by the same argument as in [1], p. 11, we see that $\mu(A)=\lim_{n\to\infty}\mu_n(A)$ is a non-negative integer.

Let K be compact, and let $G_\varepsilon=\{y;\, d(K,y)>\varepsilon\}$. Then $\partial G_\varepsilon=\{y;\, d(K,y)=\varepsilon\}$, $K=\bigcap_{\delta<\varepsilon} G_\delta$ and G_ε is represented as the disjoint union $G_\varepsilon=K\cup(\bigcup_{\delta<\varepsilon}\partial G_\delta)$. Hence there exists a sequence δ_m such that $\delta_m\to 0$, $\mu(\partial G_{\delta_m})=0$, $m\geq 1$. The relations $\mu(G_{\delta_m})=\lim_{n\to\infty}\mu_n(G_{\delta_m})$ and $\mu(K)=\lim_{m\to\infty}\mu(G_{\delta_m})$ imply that $\mu(K)$ is an integer.

If A is any bounded Borel set, then by the regularity of Borel measures in a metric space, $\mu(A)=\sup\{\mu(K);\, K\subset A,\, K \text{ is compact}\}$ is an integer. This proves the lemma.

The following lemma is a slight modification of a well-known result and will be proved easily.

Lemma 2.3. *Suppose a class $\mathfrak{C}$ of subsets of an arbitrary metric space X satisfies:* (i) *every compact set is in $\mathfrak{C}$,* (ii) *if A and B are bounded sets in $\mathfrak{C}$ and if $A\subset B$, then $B\cap A^c\in\mathfrak{C}$,* (iii) *if A and B are disjoint bounded sets in $\mathfrak{C}$, then $A\cup B\in\mathfrak{C}$,* (iv) *if $\{A_n\}$ is an increasing sequence of bounded sets in $\mathfrak{C}$, and if $A=\bigcup_n A_n$ is bounded, then $A\in\mathfrak{C}$. Then $\mathfrak{C}$ contains all bounded Borel sets.*

Lemma 2.4. *Let $\mathfrak{A}$ be the class of all Borel subsets of S. For any non-negative $\mathfrak{B}$-measurable real function φ on X, the mapping $\mu\to(\varphi,\mu)$ from S to the closed half line* $[0,\infty]$ *is $\mathfrak{A}$-measurable. In particular, for any $A\in\mathfrak{B}$, the mapping $\mu\to\mu(A)$ from S to* $[0,\infty]$ *is $\mathfrak{A}$-measurable.*

Proof. Let $\mathfrak{C}$ be the class of sets $A\in\mathfrak{B}$ such that the mapping $\mu\to\mu(A)$ is $\mathfrak{A}$-measurable. If K is compact, then $K\in\mathfrak{C}$. In fact, there exists a sequence $\{\varphi_n\}$, $\varphi_n\in C_0^+(X)$ such that $0\leq\varphi_n(x)\leq 1$, $\lim_{n\to\infty}\varphi_n(x)=\chi_K(x)$ where χ_K is the indicator function of the set K. Then for $\mu\in S$, $\lim_{n\to\infty}(\varphi_n,\mu)=\mu(K)$. The mapping $\mu\to(\varphi_n,\mu)$ are continuous. Hence $\mu\to\mu(K)$ is measurable. The remaining conditions of Lemma 2.3 are easily verified for $\mathfrak{C}$, and therefore $\mathfrak{C}$ contains every bounded Borel set. Since every $A\in\mathfrak{B}$ is the limit of an increasing sequence of bounded Borel sets, $\mathfrak{B}=\mathfrak{C}$. This proves the last half of the lemma. The first half is a consequence of the application of a standard argument to the last half.

Remark. Let $\mathfrak{C}$ be a class of subsets of X which is closed under the formation of finite intersections, and assume that $\mathfrak{C}$ generates the σ-field $\mathfrak{B}$. For instance the class of all compact sets satisfies these conditions since X is separable.

Then the argument similar to the preceding proof shows that $\mathfrak{A}$ is the smallest σ-field with respect to which all mappings $\mu\to\mu(A)$, $A\in\mathfrak{C}$ are measurable.

Definition 2.1. *A random measure P on X is a probability measure on $\mathfrak{A}$. A point process P on X is a random measure on X such that $P(S^p)=1$.*

Definition 2.2. *We say a sequence of random measures $\{P_n\}$ on X converges to a random measure P, and write $P_n\Longrightarrow P$, if probability measures P_n converges weakly to P, i.e., $\int_S f dP_n \to \int_S f dP$ for every bounded continuous real fuuction f on S.*

The following two lemmas are the uniqueness theorem and the continuity theorem of Laplace transforms in higher dimensions. In the one-dimensional case these lemmas are well-known (see [7]). The general case can be proved by reducing it to the one-dimensional case.

Lemma 2.5. *Let p and q be two probability measures in R^{k+}. If*

$$\int_{R^{k+}} e^{-(t,x)}p(dx)=\int_{R^{k+}} e^{-(t,x)}q(dx)\,, \qquad t\in R^{k+}\,, \tag{2.1}$$

where $(t,x)=t_1x_1+\cdots+t_kx_k$, R^{k+} is the positive octant $\{(x_1,\cdots,x_k);\, x_i\geq 0\}$ of R^k, then $p=q$.

Lemma 2.6. *Let $\{p_n\}$ be a sequence of probability measures in R^{k+}. If*

$$l(t)=\lim_{n\to\infty}\int_{R^{k+}} e^{-(t,x)}p_n(dx)\,, \qquad t\in R^{k+}\,, \tag{2.2}$$

exists, and if $\lim_{t\to+0} l(t)=1$, then p_n converges weakly to a probability measure p in R^{k+}, and

$$l(t)=\int_{R^{k+}} e^{-(t,x)}p(dx)\,.$$

Remark. These two lemmas are valid if (2.1) and (2.2) hold for t in a neighborhood of the origin.

Theorem 2.1. *For a random measure P on X, define a functional $l(\cdot\,;P)$ on the class of all non-negative measurable functions φ on X by*

$$l(\varphi;P)=\int_S e^{-(\varphi,\mu)}P(d\mu)\,. \tag{2.3}$$

If P and Q are two random measures on X such that

(2.4) $$l(\varphi;\ P)=l(\varphi;\ Q)\ ,\qquad \varphi\in C_0^+(X)\ ,$$

then $P=Q$.

Proof. Let $\varphi=\sum_{i=1}^{k} t_i\alpha_i$, $t_i\geq 0$. Then by Lemma 2.5, $P\tau^{-1}\pi_k^{-1}=Q\tau^{-1}\pi_k^{-1}$, $k\geq 1$, where π_k is the natural projection from R^∞ to R^k defined by $\pi_k(\omega)=(\omega_1,\cdots,\omega_k)$. This implies $P\tau^{-1}=Q\tau^{-1}$, and therefore $P=Q$ ([1] p. 19, p. 39).

Theorem 2.2. *Let* P_n *be random measures on* X. *If* P_n *converges to a random measure* P *on* X, *then*

(2.5) $$\lim_{n\to\infty} l(\varphi;\ P_n)=l(\varphi;\ P)\ ,\qquad \varphi\in C_0^+(X)\ .$$

Conversely if

(2.6) $$l(\varphi)=\lim_{n\to\infty} l(\varphi;\ P_n)\ ,$$

exists for every $\varphi\in C_0^+(X)$ *and if* $\lim_{t\to+0} l(t\varphi)=1$, *then there exists a unique random measure* P *on* X *such that* $P_n\Longrightarrow P$ *and* $l(\varphi;\ P)=l(\varphi)$.

Proof. The first half is obvious, since for $\varphi\in C_0^+(X)$ the function $\mu\to e^{-(\varphi,\mu)}$ is bounded and continuous on S. Let us prove the second half. Let $t=(t_1,\cdots,t_k)\in R^{k+}$. By the assumption we have $\lim_{t\to 0} l(t_1\alpha_1+\cdots+t_k\alpha_k)=1$. It follows from Lemma 2.6 that for each k the probability measures $P_n\tau^{-1}\pi_k^{-1}$ on R^k converges weakly to a probability measure, and therefore there exists a probability measure P' on R^∞ such that $P_n\tau^{-1}\Longrightarrow P'$ ([1] p. 19). Since $\tau(S)$ is closed in R^∞, $P'\tau(S)\geq \limsup_{n\to\infty} P_n\tau^{-1}(\tau(S))=\limsup_{n\to\infty} P_n(S)=1$. Thus $P'\tau(S)=1$, and $P=P'\tau$ is a random measure on X. If f is a bounded continuous real function on S, then so is the function $f\tau^{-1}$ on the closed set $\tau(S)$. By Tietze's theorem ([11] p. 242) there exists a bounded continuous extension g of $f\tau^{-1}$ to R^∞, and we have

$$\lim_{n\to\infty}\int_S f dP_n=\lim_{n\to\infty}\int_{R^\infty} g dP_n\tau^{-1}=\int_{R^\infty} g dP'=\int_S f dP\ .$$

This proves $P_n\Longrightarrow P$.

Remark. By the remark following Lemma 2.6, Theorem 2.1 and the last half of Theorem 2.2 are valid if (2.5) and (2.6) hold only for $\varphi\in C_0^+(X)$ such that $0\leq\varphi<1$.

3. Existence theorems. In this section we state and prove two theorems which assert the existence of random measures and point processes. These theorems seem to be included in recent results of *Harris* [9] and [10].

Lemma 3.1. *If X is a separable metric space, then for every $\delta>0$ there is a finite or infinite sequence $\{x_n\}$, $x_n\in X$, such that* (i) $d(x_m, x_n)>\delta/2$ *if* $m\neq n$, *and* (ii) $\bigcup_n S(x_n, \delta)=X$, *where* $S(x, \delta)=\{y;\ d(x, y)<\delta\}$.

Proof. Let $\{y_k\}$ be a countable dense subset of X, and let $x_1=y_1$. Suppose x_j, $j<n$, have been chosen, and let x_n be the first y_k such that $y_k\notin\bigcup_{j=1}^{n-1} S(x_j, \delta/2)$. It is easy to see that the sequence $\{x_n\}$ has the stated properties.

Theorem 3.1. *Assume that for each $n\geq 1$, disjoint bounded sets $A_i\in\mathfrak{B}$, and $t_i\geq 0$, $1\leq i\leq n$, there corresponds a real $L(t_1, \cdots, t_n;\ A_1, \cdots, A_n)$ satisfying the following properties*:

(i) *for each family of disjoint bounded $A_i\in\mathfrak{B}$, $1\leq i\leq n$, $L(\cdot, \cdots, \cdot;\ A_1, \cdots, A_n)$ is the Laplace transform of a probability distribution on R^{n+}*,

(ii) *if $(i_1, \cdots, i_n)$ is a permutation of $(1, \cdots, n)$, then*

$$L(t_{i_1}, \cdots, t_{i_n};\ A_{i_1}, \cdots, A_{i_n})=L(t_1, \cdots, t_n;\ A_1, \cdots, A_n)\ ,$$

(iii) $$L(t_1, \cdots, t_n, 0;\ A_1, \cdots, A_n, A_{n+1})=L(t_1, \cdots, t_n;\ A_1, \cdots, A_n)\ ,$$

(iv) $$\begin{aligned}&L(t_1, \cdots, t_{n-1}, t_n, t_n;\ A_1, \cdots, A_{n-1}, A_n, A_{n+1})\\&=L(t_1, \cdots, t_{n-1}, t_n;\ A_1, \cdots, A_{n-1}, A_n\cup A_{n+1})\ ,\end{aligned}$$

(v) *if $A_1\supset A_2\supset\cdots$, $\bigcap_n A_n=\phi$, then $\lim_{n\to\infty} L(t;\ A_n)=1$ for $t\geq 0$.*

Then there exists a unique random measure P on X such that

$$\int_S \exp[-\sum_{j=1}^n t_j\mu(A_j)]dP=L(t_1, \cdots, t_n;\ A_1, \cdots, A_n)\ , \tag{3.1}$$

for each $n\geq 1$, disjoint bounded $A_i\in\mathfrak{B}$, $t_i\geq 0$, $1\leq i\leq n$.

Proof. First let us prove the following inequality:

$$|L(t_1, \cdots, t_l;\ A_1, \cdots, A_l)-L(t'_1, \cdots, t'_m;\ A'_1, \cdots, A'_m)|\leq 1-L(\varepsilon;\ A)\ , \tag{3.2}$$

where $\{A_1, \cdots, A_l\}$ and $\{A'_1, \cdots, A'_m\}$, $A_i\in\mathfrak{B}$, $A'_j\in\mathfrak{B}$, $A_i\cap A_j=\phi$, $A'_i\cap A'_j=\phi$ if $i\neq j$, $\bigcup_{i=1}^l A_i=\bigcup_{j=1}^m A'_j=A$, are two partitions of a bounded set $A\in B$, and $t_i, t'_j\geq 0$, $|t_i-t'_j|<\varepsilon$ if $A_i\cap A'_j\neq\phi$, $1\leq i\leq l$, $1\leq j\leq m$. In fact, let $A''_1, \cdots, A''_n$ be an enumeration of non-void $A_i\cap A'_j$, and let $u_k=t_i$, $u'_k=t'_j$ if $A''_k=A_i\cap A'_j$. Then $|u_k-u'_k|<\varepsilon$, $1\leq k\leq n$. If $\xi_1, \cdots, \xi_n$, $\xi_k\geq 0$, are random variables on a probability space, whose joint Laplace transform is $L(t_1, \cdots, t_n;\ A''_1, \cdots, A''_n)$, then by (iv) the random variable $\sum_{k=1}^{n} \xi_k$ has the Laplace transform $L(t; A)$. It follows from (iii) and (iv) that the left hand side of (3.2) is equal to

$$|L(u_1, \cdots, u_n; A_1'', \cdots, A_n'')-L(u_1', \cdots, u_n'; A_1'', \cdots, A_n'')|$$
$$=|E[\exp(-\sum_{k=1}^{n} u_k\xi_k)-\exp(-\sum_{k=1}^{n} u_k'\xi_k)]| ,$$

which is dominated by $E[1-\exp(-\sum_{k=1}^{n} |u_k-u_k'|\xi_k)]=1-L(\varepsilon; A)$.

By Lemma 3.1, we can find for each fixed n a countable discrete subset $C_n=\{x_{nk}; k\geq 1\}$ of X such that $\bigcup_k S(x_{nk}; n^{-1})=X$. Let $A_{nk}'=S(x_{nk}; n^{-1})\cap(\bigcup_{j=1}^{k-1} S(x_{nk}; n^{-1}))^c$, and let $A_{nk}=A_{nk}'$ if A_{nk}' is bounded, and $A_{nk}=\phi$ otherwise. For each $k\geq 1$, the Laplace transform $L(t_1, \cdots, t_k; A_{n1}, \cdots, A_{nk})$ determines a unique probability measure p_{nk} on R^{k+}. It follows from (ii) and (iii) that the probability measures p_{nk}, $k=1, 2, \cdots$, are consistent, and therefore there exists a unique probability measure P_n' on $R^{\infty+}$ such that $P_n'\pi_k^{-1}=p_{nk}$. Let σ_n be the mapping from $R^{\infty+}$ to S_{C_n} defined by $\sigma_n(\omega)=\mu$ if $\mu(\{x_{ni}\})=\omega_i$. It is easy to see that $P_n=P_n'\sigma_n^{-1}$ is a probability on $\mathfrak{B}$ such that $P_n(S_{C_n})=1$.

We shall now prove that P_n converges to a random measure on X. If $\varphi\in C_0^+(X)$, then there exists a bounded open set U such that $U\supset \operatorname{supp}[\varphi]$. Let $n>2/\delta$ be fixed for a moment, where $\delta=d(\operatorname{supp}[\varphi], U^c)$. It is easy to see that except for a finite number of j, A_{nj} does not meet with $\operatorname{supp}[\varphi]$, and if $A_{nj}'\cap \operatorname{supp}[\varphi]\neq\phi$, then A_{nj}' is bounded, i.e., $A_{nj}=A_{nj}'$ and $A_{nj}\subset U$. Let $B_{n1}, \cdots, B_{nl}$, where l depends on n, be an enumeration of A_{nj}, $j\geq 1$, such that $A_{nj}\cap \operatorname{supp}[\varphi]\neq\phi$, and let $y_{ni}=x_{nj}$ if $B_{ni}=A_{nj}$. Then $\operatorname{supp}[\varphi]\subset \bigcup_{i=1}^{l} B_{ni}\subset K$, where $K=\bar{U}$ is a compact set independent of n.

$$\int_S e^{-(\varphi,\mu)}P_n(d\mu)=L(\varphi(y_{n1}), \cdots, \varphi(y_{nl}), 0; B_{n1}, \cdots, B_{nl}, K\cap(\bigcup_{i=1}^{l} B_{ni})^c) . \tag{3.3}$$

For any $\varepsilon>0$ there exists $n>2/\delta$ such that $d(x, y)<1/n$ implies $|\varphi(x)-\varphi(y)|<\varepsilon$. If $m>n$, then by (3.2) and (3.3) $\left|\int_S e^{-(\varphi,\mu)}dP_n-\int_S e^{-(\varphi,\mu)}dP_m\right|$ is dominated by $1-L(\varepsilon; K)$ which tends to 0 with ε. Hence $l(\varphi)=\lim_n l(\varphi; P_n)$ exists for every $\varphi\in C_0^+(X)$. Since

$$l(t\varphi; P_n)=L(t\varphi(y_{n1}), \cdots, t\varphi(y_{nl}), 0; B_{n1}, \cdots, B_{nl}, K\cap(\bigcup_{i=1}^{l} B_{ni})^c)$$
$$\geq L(t\cdot\sup_{x\in X}\varphi(x); K) , \qquad n\geq 1 ,$$

we have $\lim_{t\to+0} l(t\varphi)=1$. It follows from Theorem 2.2 that there exists a random measure P on X such that $P_n \Longrightarrow P$, $l(\varphi; P)=l(\varphi)$.

It remains to prove that P satisfies (3.1). Let $K_1, \cdots, K_n$ be disjoint compact subsets of X. Then for each $j\geq 1$, there exists disjoint bounded open sets G_{ij}, $1\leq i\leq n$, such that $G_{ij}\supset K_i$, $1\leq i\leq n$, and $G_{i1}\supset G_{i2}\supset\cdots$, $\bigcap_j G_{ij}=K_i$. Let $\varphi_{ij}\in C_0^+(X)$, $1\leq i\leq n$, $j\geq 1$, be such that $0\leq\varphi_{ij}(x)\leq 1$, $\varphi_{ij}(x)=0$ if $x\notin G_{ij}$, $\varphi_{ij}(x)=1$ if $x\in K_i$. Clearly $\lim_{j\to\infty}\varphi_{ij}(x)=\chi_{K_i}(x)$. It follows from bounded convergence theorem and from (v) that

$$\begin{aligned}\int_S \exp\Big[-\sum_{i=1}^n t_i\mu(K_i)\Big]P(d\mu)&=\lim_{j\to\infty}\int_S \exp\Big[-\Big(\sum_{i=1}^n t_i\varphi_{ij},\mu\Big)\Big]P(d\mu)\\ &\geq \limsup_{j\to\infty} L(t_1,\cdots,t_n; G_{1j},\cdots,G_{nj})\\ &=L(t_1,\cdots,t_n;K_1,\cdots,K_n)+\limsup_{j\to\infty} L(t_1,\cdots,t_n; G_{1j}\cap K_1^c,\cdots,G_{nj}\cap K_n^c)\\ &\geq L(t_1,\cdots,t_n;K_1,\cdots,K_n)+\limsup_{j\to\infty} L\Big(\max_{1\leq i\leq n} t_i; \bigcup_{i=1}^n (G_{ij}\cap K_i^c)\Big)\\ &=L(t_1,\cdots,t_n;K_1,\cdots,K_n)\ .\end{aligned}$$

The reverse inequality is easily verified and we have proved (3.1) when A_i's are all compact. Let $n\geq 1$, and bounded disjoint sets $A_2,\cdots,A_n\in\mathfrak{B}$ be fixed for a moment. Assume that (3.1) holds for each compact A_1 and these fixed sets A_2, $\cdots, A_n$. Let $\mathfrak{C}=\mathfrak{C}(A_2,\cdots,A_n)$ be the class of bounded sets $A_1\in\mathfrak{B}$ for which (3.1) holds. By (iv) and (v) $\mathfrak{C}$ satisfies the conditions of Lemma 2.3, and therefore every bounded Borel set is in $\mathfrak{C}$. Now starting from compact $A_2,\cdots,A_n$ and using (ii), we see that (3.1) holds for every family of disjoint bounded Borel sets. This completes the proof.

Theorem 3.2. *Assume that for each integer $n\geq 1$, disjoint bounded sets A_i, and real s_i, $|s_i|\leq 1$, $1\leq i\leq n$, there corresponds a real number $\Phi(s_1,\cdots,s_n; A_1,\cdots, A_n)$ satisfying the following properties:*

(i) *for each family of disjoint bounded $A_i\in\mathfrak{B}$, $1\leq i\leq n$, $\Phi(\cdot,\cdots,\cdot; A_1,\cdots,A_n)$ is the probability generating function of a probability distribution concentrated on the lattice points of R^{n+},*

(ii) *if $(i_1,\cdots,i_n)$ is a permutation of $(1,\cdots,n)$, then*

$$\Phi(s_{i_1},\cdots,s_{i_n}; A_{i_1},\cdots,A_{i_n})=\Phi(s_1,\cdots,s_n;A_1,\cdots,A_n)\ ,$$

(iii) $$\Phi(s_1,\cdots,s_n,1;A_1,\cdots,A_n,A_{n+1})=\Phi(s_1,\cdots,s_n;A_1,\cdots,A_n)\ ,$$

(iv) $$\begin{aligned}&\Phi(s_1,\cdots,s_{n-1},s_n,s_n;A_1,\cdots,A_{n-1},A_n,A_{n+1})\\ &\quad=\Phi(s_1,\cdots,s_{n-1},s_n;A_1,\cdots,A_{n-1},A_n\cup A_{n+1})\ ,\end{aligned}$$

(v) *if* $A_1 \supset A_2 \supset \cdots$, $\bigcap_n A_n = \phi$, *then* $\lim_{n\to\infty} \Phi(s; A_n) = 1$ *for* $|s| \leq 1$.

Then there exists a unique point process P on X such that

$$\int_S s_1^{\mu(A_1)} \cdots s_n^{\mu(A_n)} dP = \Phi(s_1, \cdots, s_n; A_1, \cdots A_n) , \tag{3.4}$$

for each $n \geq 1$, *disjoint bounded* $A_i \in \mathfrak{B}$, $|s_i| \leq 1$, $1 \leq i \leq n$.

Proof. The function L defined by

$$L(t_1, \cdots, t_n; A_1, \cdots, A_n) = \Phi(e^{-t_1}, \cdots, e^{-t_n}; A_1, \cdots, A_n) ,$$

satisfies all conditions of Theorem 2.1. Therefore there exists a unique random measure P on X for which (3.1) and therefore (3.4) holds for every disjoint bounded $A_i \in \mathfrak{B}$. Let P_n be random measures defined in the proof of Theorem 3.1. Clearly $P_n(S^p) = 1$, $n \geq 1$. Since S^p is closed in S and since $P_n \Longrightarrow P$, we have $P(S^p) = 1$, this proves the theorem.

4. Mixed Poisson processes. In this section we prove the existence of a mixed Poisson process $\mathfrak{M}(P)$ whose "mixture" process is an arbitrary random measure P, and give some relations between P and $\mathfrak{M}(P)$.

Theorem 4.1. *Let P be a random measure on X. Then there exists a unique point process* $Q = \mathfrak{M}(P)$ *on X such that*

$$\int_S \prod_{j=1}^n s_j^{\mu(A_j)} Q(d\mu) = \int_S \exp\left[\sum_{j=1}^n (s_j - 1)\cdot \mu(A_j)\right] P(d\mu) , \tag{4.1}$$

for every $n \geq 1$, *disjoint bounded* $A_j \in \mathfrak{B}$, $|s_j| \leq 1$, $1 \leq j \leq n$, *and*

$$l(\varphi; Q) = l(1 - e^{-\varphi}; P) , \tag{4.2}$$

for every $\varphi \in C_0^+(X)$.

Proof. For any finite family of disjoint bounded sets $A_j \in \mathfrak{B}$,

$$\Phi(s_1, \cdots, s_n; A_1, \cdots, A_n) = \int_S \exp\left[\sum_{j=1}^n (s_j - 1)\cdot \mu(A_j)\right] P(d\mu)$$

is a probability generating function. The family of functions Φ satisfies the conditions of Theorem 3.2, and therefore there exists a unique point process Q satisfying (4.1). The relation (4.2) is proved by a standard approximation procedure.

Definition 4.1. *The point process* $Q = \mathfrak{M}(P)$ *is called the mixed Poisson process generated by the random measure P. If* $P(\{\mu\}) = 1$, $\mu \in S$, *then* $Q = \mathfrak{M}(P)$ *is called the Poisson process generated by the measure* μ.

Theorem 4.2. *If P and Q are two random measures such that $\mathfrak{M}(P)=\mathfrak{M}(Q)$, then $P=Q$.*

Proof. This follows from (4.2), Theorem 2.1 and the remark following Theorem 2.2.

Theorem 4.3. *If P is a random measure and if $Q=\mathfrak{M}(P)$, then for every φ and ψ in $C_0^+(X)$*

$$\int_S (\varphi, \mu)dQ=\int_S (\varphi, \mu)dP\,, \tag{4.3}$$

and

$$\int_S (\varphi, \mu)(\psi, \mu)dQ=\int_S (\varphi, \mu)(\psi, \mu)dP+\int_S (\varphi\psi, \mu)dP\,, \tag{4.4}$$

where both sides of (4.3) *or* (4.4) *may be infinite.*

Proof. By a well-known property of Laplace transforms we have

$$\begin{aligned}\int_S (\varphi, \mu)dQ &= -\lim_{t\to+0}\frac{d}{dt}l(t\varphi;\, Q)=-\lim_{t\to+0}\frac{d}{dt}l(1-e^{-t\varphi};\, P)\\ &= -\lim_{t\to+0}\frac{d}{dt}\int_S \exp[-(1-e^{-t\varphi}, \mu)]P(d\mu)\\ &= \lim_{t\to+0}\int_S\left[\int_X \varphi e^{-t\varphi}d\mu\right]\exp\left[-\int_X (1-e^{-t\varphi})\,d\mu\right]P(d\mu)\\ &= \int_S (\varphi, \mu)P(d\mu)\,.\end{aligned}$$

The exchange of differentiation and integration is justified since $(\varphi e^{-t\varphi}, \mu)\exp[-(1-e^{-t\varphi}, \mu)]$ is dominated by an integrable function independent of $t>t_0>0$. The last equality follows from the monotone convergence theorem. Similarly we have

$$\begin{aligned}\int_S (\varphi, \mu)(\psi, \mu)dQ &= \lim_{t,u\to+0}\frac{\partial^2}{\partial t\partial u}l(1-e^{-t\varphi-u\psi};\, P)\\ &= \lim_{t,u\to+0}\int_S [(\varphi\psi\, e^{-t\varphi-u\psi}, \mu)+(\varphi\, e^{-t\varphi-u\psi}, \mu)(\psi\, e^{-t\varphi-u\psi}, \mu)]\\ &\qquad\cdot\exp[-(1-e^{-t\varphi-u\psi}, \mu)]P(d\mu)\\ &= \int_S [(\varphi\psi, \mu)+(\varphi, \mu)(\psi, \mu)]P(d\mu)\,.\end{aligned}$$

Theorem 4.4. *Let $Q_n=\mathfrak{M}(P_n)$ be the mixed Poisson processes generated by random measures P_n. In order that Q_n converge to a point process Q, it is necessary and sufficient that P_n converge to a random measure P. In this case $Q=\mathfrak{M}(P)$.*

Proof. From (4.2)

$$l(\varphi;\, Q_n)=l(1-e^{-\varphi};\, P_n)\ .$$

If $Q_n \Longrightarrow Q$, then for every $\psi \in C_0^+(X)$ such that $\psi(x)<1$, we have

$$l(\psi)=\lim_{n\to\infty} l(\psi;\, P_n)=\lim_{n\to\infty} l(-\log(1-\psi);\, Q_n)=l(-\log(1-\psi);\, Q)\ ,$$

and

$$\lim_{t\to+0} l(t\psi)=\lim_{t\to+0} l(-\log(1-t\psi);\, Q)=1\ .$$

It follows from Theorem 2.2 and the remark following it that there exists a unique random measure P such that $P_n \Longrightarrow P$ and $l(1-e^{-\varphi};\, P)=l(\varphi;\, Q)$. The converse part is proved similarly.

5. **Random translations.** Let λ be a substochastic transition function on the measurable space $(X, \mathfrak{B})$, i.e., λ is a function on $X\times\mathfrak{B}$ satisfying: (i) for fixed $x\in X$, $\lambda(x,\cdot)$ is a measure on $\mathfrak{B}$ such that $\lambda(x, X)\leq 1$, and (ii) for fixed $A\in\mathfrak{B}$, $\lambda(\cdot, A)$ is a $\mathfrak{B}$-measurable function on X. For real bounded measurable function φ on X, and for $\mu\in S$ we write

$$T_\lambda\varphi(x)=\int_X \varphi(y)\lambda(x, dy)\ , \qquad x\in X\ ,$$

and

$$U_\lambda\mu(A)=\int_X \lambda(x, A)\mu(dx)\ , \qquad A\in\mathfrak{B}\ ,$$

Note that $U_\lambda\mu$ is a measure on $\mathfrak{B}$ which may assume the value $+\infty$ for compact sets. By Lemma 2.4 the mapping $\mu\to U_\lambda\mu(A)$ is $\mathfrak{A}$-measurable for each $A\in\mathfrak{B}$.

Lemma 5.1. *Let $S_\lambda=\{\mu;\ \mu\in S,\ U_\lambda\mu\in S\}$. Then $S_\lambda\in\mathfrak{A}$.*

Proof. Choose an increasing sequence $\{A_n\}$, $A_n\in\mathfrak{B}$, such that $\bigcup_n A_n=X$. Then $S_\lambda=\{\mu;\ \mu\in S,\ U_\lambda\mu(A_n)<\infty \text{ for } n\geq 1\}\in\mathfrak{A}$.

Theorem 5.1. *Let P be a random measure on X, and λ be a substochastic transition function on $(X, \mathfrak{B})$. In order that there exist a unique random measure Q on X satisfying:*

$$\int_S \exp[-\sum_{j=1}^n t_j\mu(A_j)]Q(d\mu)=\int_S \exp[-\sum_{j=1}^n t_jU_\lambda\mu(A_j)]P(d\mu)\ , \tag{5.1}$$

for every $n\geq 1$, bounded disjoint $A_j\in\mathfrak{B}$, $t_j\geq 0$, $1\leq j\leq n$, it is necessary and sufficient *that* $P(S_\lambda)=1$. *If this is the case, then*

$$l(\varphi; Q)=l(T_\lambda\varphi; P)\ ,$$

for $\varphi\in C_0^+(X)$.

Proof. Assume $P(S_\lambda)=1$. The family of functions

$$L(t_1,\cdots,t_n; A_1,\cdots,A_n)=\int_S \exp\left[-\sum_{j=1}^n t_j U_\lambda\mu(A_j)\right]P(d\mu)\ ,$$

satisfies the conditions of Theorem 3.1, and therefore determines a unique random measure Q on X. The relation (5.2) may be proved by a standard argument. If $P(S_\lambda)<1$, then $P\{\mu;\ U_\lambda\mu(A)=\infty\}>0$ for some bounded $A\in\mathfrak{B}$, and the right hand side of (5.1) with $n=1$, $A_1=A$, cannot be the Laplace transform of a probability distribution.

Definition 5.1. *When* $P(S_\lambda)=1$, *the random measure* Q *on* X *defined in Theorem 5.1 is called the translation of the random measure* P *by the transition function* λ, *and is denoted by* $\mathfrak{T}_\lambda(P)$.

Theorem 5.2. *Let* P *be a point process on* X, *and* λ *be a substochastic transition function on* $(X,\mathfrak{B})$. *In order that there exist a point process* Q *such that for* $n\geq 1$, *disjoint bounded* A_j, $|s_j|\leq 1$, $1\leq j\leq n$,

$$\int_S \prod_{j=1}^n s_j^{\mu(A_j)}Q(d\mu)=\int_S \exp\left(\log\left[1-\sum_{j=1}^n(1-s_j)\lambda(x, A_j)\right], \mu\right)P(d\mu)\ , \tag{5.3}$$

it is necessary and sufficient that $P(S_\lambda)=1$. *If this is the case, then for* $\varphi\in C_0^+(X)$,

$$l(\varphi; Q)=l(-\log(1-T_\lambda(1-e^{-\varphi})); P)\ . \tag{5.4}$$

Proof. For every $\mu\in S^p$, let $\{x_k; k\geq 1\}$ be the support of μ and let $n_k=\mu(\{x_k\})$. Then

$$\exp\left(\log\left[1-\sum_{j=1}^n(1-s_j)\lambda(x, A_j)\right], \mu\right)=\prod_k\left[1-\sum_{j=1}^n(1-s_j)\lambda(x_k, A_j)\right]^{n_k} \tag{5.5}$$

is a probability generating function if and only if this infinite product converges, or equivalently

$$\sum_j\sum_k n_k\lambda(x_k, A_j)=\sum_j U_\lambda\mu(A_j)<+\infty\ .$$

Thus if $\mu\in S^p\cap S_\lambda$, then (5.5) is a generating function for any finite family of disjoint bounded $A_j\in\mathfrak{B}$. Hence $P(S_\lambda)=1$ implies that the right hand side $\Phi(s_1,\cdots,s_n; A_1,\cdots,A_n)$ of (5.3) is a probability generating function. It is easy to verify that the family of functions $\Phi(s_1,\cdots,s_n; A_1,\cdots,A_n)$ satisfies the conditions of Theorem 3.2, and therefore there exists a unique point process Q satisfying (5.3). The relation (5.4) follows from (5.3).

If $P(S_\lambda)<1$, then there exists a bounded set $A\in\mathfrak{B}$ such that $P\{\mu;\ U_\lambda\mu(A)=\infty\}>0$. It follows that

$$\int_S \exp(\log[1-(1-s)\lambda(x,A)],\mu)P(d\mu)\ , \tag{5.6}$$

is dominated by $P\{\mu;\ U_\lambda\mu(A)<\infty\}<1$ for every s. This shows that (5.6) is not a generating function.

Definition 5.2. *If P is a point process on X satisfying $P(S_\lambda)=1$, then the point process Q defined in Theorem 5.2 is called the random translation of P by the transition function λ, and is denoted by $\mathfrak{R}_\lambda(P)$.*

Remark. The intuitive definition of random translation is as follows. A point process determines stochastically a "particle system" on X. We move each "particle" of this system independently according to the transition function λ, and get a new particle system which is the random translation of the initial point process. The quantity $1-\lambda(x,X)$ is the probability that a particle at the position x "disappears".

Theorem 5.3. *Let λ_1 and λ_2 be two substochastic transition functions on $(X,\mathfrak{B})$, and let*

$$\lambda(x,A)=\int_X \lambda_1(x,dy)\lambda_2(y,A)\ , \qquad A\in\mathfrak{B}\ .$$

Then the translation $\mathfrak{T}_\lambda(P)$ of a random measure P (or the random translation $\mathfrak{R}_\lambda(P)$ of a point process P) exists if and only if $\mathfrak{T}_{\lambda_2}(\mathfrak{T}_{\lambda_1}(P))$ (or $\mathfrak{R}_{\lambda_2}(\mathfrak{R}_{\lambda_1}(P))$) exists, and

$$\mathfrak{T}_{\lambda_2}(\mathfrak{T}_{\lambda_1}(P))=\mathfrak{T}_\lambda(P)\ , \tag{5.7}$$

$$\mathfrak{R}_{\lambda_2}(\mathfrak{R}_{\lambda_1}(P))=\mathfrak{R}_\lambda(P)\ . \tag{5.8}$$

Proof. We prove (5.7) only since (5.8) is similarly proved. Suppose that $\mathfrak{T}_{\lambda_2}(\mathfrak{T}_{\lambda_1}(P))$ and $\mathfrak{T}_\lambda(P)$ are both defined, then since $T_{\lambda_1}(T_{\lambda_2}\varphi)=T_\lambda\varphi$, and since (5.2) holds for any non-negative continuous φ, we have

$$l(\varphi;\ \mathfrak{T}_{\lambda_2}(\mathfrak{T}_{\lambda_1}(P)))=l(T_{\lambda_2}\varphi;\ \mathfrak{T}_{\lambda_1}(P))=l(T_{\lambda_1}(T_{\lambda_2}\varphi);\ P)=l(T_\lambda\varphi;\ P)=l(\varphi;\ \mathfrak{T}_\lambda(P))\ .$$

Thus by Theorem 2.1 we have (5.7). Moreover these equalities can be used to prove that $\mathfrak{T}_{\lambda_2}(\mathfrak{T}_{\lambda_1}(P))$ and $\mathfrak{T}_\lambda(P)$ exist simultaneously.

Theorem 5.4. *Let P be a random measure on X and let λ be a substochastic transition function on $(X, \mathfrak{B})$. Then $P(S_\lambda)=1$ if and only if $\mathfrak{M}(P)(S_\lambda)=1$, and*

$$\mathfrak{R}_\lambda(\mathfrak{M}(P))=\mathfrak{M}(\mathfrak{T}_\lambda(P)) . \tag{5.9}$$

Proof. If $P(S_\lambda)=1$ and $\mathfrak{M}(P)(S_\lambda)=1$, then from (4.2), (5.2), and (5.4), we have

$$\begin{aligned} l(\varphi; \mathfrak{M}(\mathfrak{T}_\lambda(P))) &= l(1-e^{-\varphi}; \mathfrak{T}_\lambda(P))=l(T_\lambda(1-e^{-\varphi}); P) \\ &= l(-\log(1-T_\lambda(1-e^{-\varphi})); \mathfrak{M}(P))=l(\varphi; \mathfrak{R}_\lambda(\mathfrak{M}(P))) , \end{aligned}$$

where we used the fact that (4.2) holds for any non-negative continuous φ. These equalities can also be used to show the equivalence of $P(S_\lambda)=1$ and $\mathfrak{M}(P)(S_\lambda)=1$.

Theorem 5.5. *Let X be not compact, P a point process on X. Let $\{\lambda_n\}$ be a sequence of substochastic transition function on $(X, \mathfrak{B})$ such that*

$$P(S_{\lambda_n})=1 , \qquad n\geq 1 , \tag{5.10}$$

and for every bounded $A\in\mathfrak{B}$,

$$\lim_{n\to\infty} \sup_{x\in X} \lambda_n(x, A)=0 . \tag{5.11}$$

In order that $\mathfrak{R}_{\lambda_n}(P)$ converge to a point process Q, it is necessary and sufficient that $\mathfrak{T}_{\lambda_n}(P)$ converge to a random measure Q'. If this is the case, then $Q=\mathfrak{M}(Q')$.

Proof. If $\varphi\in C_0^+(X)$, then $\psi=1-e^{-\varphi}\in C_0^+(X)$. Let $a_n=\sup_{x\in X} T_{\lambda_n}\psi(x)$, then it follows from (5.11) that

$$\lim_{n\to\infty} a_n=0 . \tag{5.12}$$

Put

$$A_n(\mu)=\exp[-(T_{\lambda_n}\psi, \mu)]-\exp[(\log(1-T_{\lambda_n}\psi), \mu)] .$$

It is easy to see that

$$l(\varphi; \mathfrak{M}(\mathfrak{T}_\lambda(P)))-l(\varphi; \mathfrak{R}_{\lambda_n}(P)) = \int_S A_n(\mu)P(d\mu) .$$

Using (5.12) and an elementary inequaliy $0<-x-\log(1-x)<x^2$ for small x, we have

$$\begin{aligned} 0<A_n(\mu) &= [1-\exp(\log(1-T_{\lambda_n}\psi)+T_{\lambda_n}\psi, \mu)]\exp[-(T_{\lambda_n}\psi, \mu)] \\ &\leq 1-\exp[-((T_{\lambda_n}\psi)^2, \mu)] \\ &\leq 1-\exp[-a_n(T_{\lambda_n}\psi, \mu)] \\ &\leq 1-\exp[a_n(\log(1-T_{\lambda_n}\psi), \mu)] . \end{aligned} \tag{5.13}$$

If $\mathfrak{T}_{\lambda_n}(P)$ converges to a random measure Q' and if $a_n<\varepsilon$, then by (5.13)

$$\begin{aligned}\limsup_{n\to\infty}\int_S A_n(\mu)P(d\mu) &\le 1-\lim_{n\to\infty}\int_S \exp[-a_n(T_{\lambda_n}\psi,\mu)]P(d\mu)\\ &\le 1-\lim_{n\to\infty} l(\varepsilon T_{\lambda_n}\psi;P)\\ &=1-l(\varepsilon\psi;Q'),\end{aligned}$$

which tends to 0 with ε. It follows from Theorem 4.4 that for $\varphi\in C_0^+(X)$

$$\lim_{n\to\infty} l(\varphi;\mathfrak{R}_{\lambda_n}(P))=\lim_{n\to\infty} l(\varphi;\mathfrak{M}(\mathfrak{T}_{\lambda_n}(P)))=l(\varphi;\mathfrak{M}(Q')).$$

Hence $\mathfrak{R}_{\lambda_n}(P)\Longrightarrow\mathfrak{M}(Q')$.

Conversely if $\mathfrak{R}_{\lambda_n}(P)$ converges to a point process Q, if $a_n<\varepsilon<1$, then

$$\begin{aligned}\limsup_{n\to\infty}\int_S A_n(\mu)P(d\mu) &\le 1-\lim_{n\to\infty} l(-\varepsilon\log(1-T_{\lambda_n}(1-e^{-\varphi}));P)\\ &\le 1-\lim_{n\to\infty} l(-\log(1-T_{\lambda_n}(1-e^{-\varepsilon\varphi}));P)\\ &=1-\lim_{n\to\infty} l(\varepsilon\varphi;\mathfrak{R}_{\lambda_n}(P))\\ &=1-l(\varepsilon\varphi;Q),\end{aligned}$$

which becomes arbitrarily small with ε. Hence

$$\lim_{n\to\infty} l(\varphi;\mathfrak{M}(\mathfrak{T}_{\lambda_n}(P)))=\lim_{n\to\infty} l(\varphi;\mathfrak{R}_{\lambda_n}(P))=l(\varphi;Q).$$

It follows from Theorem 4.4 that $\mathfrak{T}_{\lambda_n}(P)$ converges to a random measure Q' and $Q=\mathfrak{M}(Q')$. This completes the proof.

Theorem 5.6. *Let X be non-compact. Assume a point process P on X and a sequence $\{\lambda_n\}$ of substochastic transition functions on $(X,\mathfrak{B})$ satisfy the conditions* (5.10) *and* (5.11) *of Theorem 5.5. In order that $P_n=\mathfrak{R}_{\lambda_n}(P)$ converge to the Poisson process Q generated by a measure $\mu\in S$, it is necessary and sufficient that*

$$\lim_{n\to\infty} P\{\nu;\,|U_{\lambda_n}\nu(A)-\mu(A)|>\varepsilon\}=0, \tag{5.14}$$

for any $\varepsilon>0$ and bounded $A\in\mathfrak{B}$ such that $\mu(\partial A)=0$.

Proof. It follows from Theorem 5.5 that $P_n\Longrightarrow Q$ if and only if for $\varphi\in C_0^+(X)$ and $\varepsilon>0$

$$\lim_{n\to\infty} P\{\nu;\,|(\varphi,U_{\lambda_n}\nu)-(\varphi,\mu)|>\varepsilon\}=0. \tag{5.15}$$

If $P_n\Longrightarrow Q$, and if $A\in\mathfrak{B}$ is a bounded set such that $\mu(\partial A)=0$, then there exist $\varphi_i\in C_0^+(X)$, $i=1,2$, such that $\varphi_1\le\chi_A\le\varphi_2$, $(\varphi_2-\varphi_1,\mu)<\varepsilon/2$. It follows from (5.15) that

$$\begin{aligned}&\lim_{n\to\infty} P\{\nu;\ |U_{\lambda_n}\nu(A)-\mu(A)|>\varepsilon\}\\&\quad\le \lim_{n\to\infty} P\{\nu;\ (\varphi_1, U_{\lambda_n}\nu)-(\varphi_2,\mu)<-\varepsilon \quad\text{or}\quad (\varphi_2, U_{\lambda_n}\nu)-(\varphi_1,\mu)>\varepsilon\}\\&\quad\le \lim_{n\to\infty} P\{\nu;\ (\varphi_1, U_{\lambda_n}\nu)-(\varphi_1,\mu)<-\varepsilon/2 \quad\text{or}\quad (\varphi_2, U_{\lambda_n})-(\varphi_2,\mu)>\varepsilon/2\}\\&\quad=0\ .\end{aligned}$$

Let us prove the converse. We assume (5.14) and prove first that for any $\varepsilon>0$,

(5.16) $$\lim_{n\to\infty} P\{\nu;\ \mu(F)<U_{\lambda_n}\nu(F)-\varepsilon\}=0\ ,$$

if F is compact, and

(5.17) $$\lim_{n\to\infty} P\{\nu;\ \mu(G)>U_{\lambda_n}\nu(G)+\varepsilon\}=0\ ,$$

if G is bounded open. In fact, let $G_\delta=\{y;\ d(y,F)<\delta\}$, then for any $\varepsilon>0$ we can choose $\delta>0$ such that $\mu(\partial G_\delta)=0$, $\mu(G_\delta\cap F^c)<\varepsilon$. It follows from (5.14) that

$$\lim_{n\to\infty} P\{\nu;\ \mu(F)<U_{\lambda_n}\nu(F)-2\varepsilon\}\le \lim_{n\to\infty} P\{\nu;\ \mu(G_\delta)<U_{\lambda_n}\nu(G_\delta)-\varepsilon\}=0\ .$$

This proves (5.16). Note that for any bounded open G, we can choose a compact F such that $F\supset G$, $\mu(\partial F)=0$. It follows from (5.14) and (5.16) that

$$\begin{aligned}&\lim_{n\to\infty} P\{\nu;\ \mu(G)>U_{\lambda_n}\nu(G)+2\varepsilon\}\\&\quad\le \lim_{n\to\infty} P\{\nu;\ \mu(F)>U_{\lambda_n}\nu(F)+\varepsilon\}+\lim_{n\to\infty} P\{\nu;\ \mu(F\cap G^c)<U_{\lambda_n}\nu(F\cap G^c)-\varepsilon\}=0\ .\end{aligned}$$

This shows (5.17).

In order to prove (5.15) for $\varphi\in C_0^+(X)$, we may and do assume that $0\le\varphi(x)\le 1$. Let $k>0$ be an integer such that $\mu(K)\le k\varepsilon$, where $K=\operatorname{supp}[\varphi]$, and let F_i be the compact set $F_i=\{x;\ i/k\le\varphi(x)\}$, $0\le i\le k$. Then for every $\nu\in S$,

$$k^{-1}\sum_{i=1}^{k}\nu(F_i)\le(\varphi,\nu)\le k^{-1}\nu(K)+k^{-1}\sum_{i=1}^{k}\nu(F_i)\ .$$

It follows from (5.16) that

(5.18) $$\begin{aligned}&\lim_{n\to\infty} P\{\nu;\ (\varphi,\mu)<(\varphi, U_{\lambda_n}\nu)-3\varepsilon\}\\&\quad\le \lim_{n\to\infty} P\{\nu;\ k^{-1}\sum_{i=1}^{k}\mu(F_i)<k^{-1}\sum_{i=1}^{k}U_{\lambda_n}\nu(F_i)-\varepsilon\}\\&\qquad+\lim_{n\to\infty} P\{\nu;\ k^{-1}U_{\lambda_n}\nu(K)>\varepsilon+k^{-1}\mu(K)\}=0\ .\end{aligned}$$

Similarly we can use (5.17) to obtain

(5.19) $$\lim_{n\to\infty} P\{\nu;\ (\varphi,\mu)>(\varphi, U_{\lambda_n}\nu)+3\varepsilon\}=0\ ,$$

The relations (5.18) and (5.19) prove (5.15).

Let X be a locally compact abelian group satisfying the second axiom of countability. It is well-known ([11] p. 210) that there exists an invariant metric d for X, i.e., d is a metric such that $d(x+z, y+z)=d(x,y)$ for x,y and z in X, and the topology of X is that derived from d (the invariance will not be used). Consider a spatially homogeneous transition function λ on $(X,\mathfrak{B})$, i.e., $\lambda(x,A)=\lambda_0(A-x)$, where $A-x=\{y-x; y\in A\}$ and λ_0 is a probability measure on $(X,\mathfrak{B})$. All of our previous results apply to this case. For example we have the following corollary to Theorem 5.6. In the following statement we mean by the Poisson process with parameter $c>0$, the Poisson process generated by the measure $c\eta$, where η is a fixed Haar measure on $(X,\mathfrak{B})$.

Corollary 5.1. *Let X be a locally compact non-compact second countable abelian group. Assume that a point process P on X, and a sequence $\{\lambda_n\}$ of probability measures on $(X,\mathfrak{B})$ satisfy the following:*

(5.20) $$P\{\mu;\ \mu*\lambda_n(A)<\infty \text{ for every bounded } A\in\mathfrak{B}\}=1, \qquad n\geq 1,$$

where $\mu*\lambda_n(A)=\int_X \lambda_n(A-x)\mu(dx)$, *and*

(5.21) $$\limsup_{n\to\infty}\ \sup_{x\in X}\lambda_n(A-x)=0\ ,$$

for every bounded $A\in\mathfrak{B}$. In order that the random translation $\mathfrak{B}_{\lambda_n}(P)$ converge to the Poisson process with parameter $c>0$, it is necessary and sufficient that

$$\lim_{n\to\infty} P\{\mu;\ |\mu*\lambda_n(A)-c\eta(A)|>\varepsilon\}=0,$$

for any $\varepsilon>0$ and bounded $A\in\mathfrak{B}$ such that $\eta(\partial A)=0$.

Remark. Assume (5.20) and (5.21). Let $S_c=\{\mu;\ \mu*\lambda_n\to c\eta\}\in A$. If $P(S_c)=1$, then it is obvious that $\mathfrak{R}_{\lambda_n}(P)$ converges to the Poisson process with parameter c. However, as the following example shows, the converse is not true.

Example. Let X be the additive group of integers with counting measure. S is the class of all two-sided sequences $\mu=\{\cdots,\mu_{-1},\mu_0,\mu_1,\cdots\}$, $\mu_i\geq 0$. $\mathfrak{B}$ is the σ-field generated by Borel cylinders. Let $S_0=\{\mu;\ \mu_j=1 \text{ for } j\leq 1\}$, $S_n=\{\mu;\ \mu_j=1$ for $2^n\leq j\leq 2^{n+1}-1\}$, $S_n'=\{\mu;\ \mu_j=0$ for $2^n\leq j\leq 2^{n+1}-2$, $\mu_j=2^n$ for $j=2^{n+1}-1\}$. Let P be the point process such that $P(S_0)=1$, $P(S_n)=1-n^{-1}$, $P(S_n')=n^{-1}$, and the events S_n, $n\geq 1$ are independent. Let λ_n be such that $\lambda_n(\{j\})=n^{-1}$, $1\leq j\leq n$. Then $\mathfrak{R}_{\lambda_n}(P)$ converges to the Poisson process with parameter 1. However it follows from Borel-Cantelli lemma that for every j, $\mu*\lambda_n(\{j\})$ does not converge with probability one.

6. Point processes invariant under random translation. In this section we consider point processes invariant under a given random translation. The following theorem is a generalization of a result of *Derman* [4]. For related results, see *Brown* [2].

Theorem 6.1. *Let $P=\mathfrak{M}(Q)$ be a point process on X generated by a random measure Q, and let λ be a substochastic transition function on $(X, \mathfrak{B})$. In order that $\mathfrak{R}_\lambda(P)=P$, it is necessary and sufficient that $\mathfrak{T}_\lambda(Q)=Q$. In particular, in order that the random translation $\mathfrak{R}_\lambda(P)$ of the Poisson process P generated by a measure $\mu\in S$ coincide with P, it is necessary and sufficient that*

$$\mu(A)=\int_X \lambda(x, A)\mu(dx)\ ,$$

for every bounded $A\in\mathfrak{B}$.

Proof. Immediate from Theorem 4.2 and Theorem 5.3.

In what follows X is a locally compact second countable abelian group with a fixed Haar measure η. Let $S^H=\{\mu;\ \mu=c\eta, c>0\}$ denote the class of all Haar measures on $(X, \mathfrak{B})$. It is easily proved that $S^H\in\mathfrak{A}$.

Definition 6.1. *If $Q=\mathfrak{M}(P)$ is a mixed Poisson process on X and if $P(S^H)=1$, then Q is called a mixed Poisson process in the strict sense.*

Lemma 6.1. *Let X be a topological abelian group, and let $M\subset X$. In order that the product group $X\times X=\{(x, y);\ x, y\in X\}$ be generated by the product set $M\times M$, it is necessary and sufficient that X be generated by $x+M$ for every $x\in X$.*

Proof. To prove the necessity, assume that for some $x\in X$, $x+M$ is contained in a proper closed subgroup Y of X. Since $(y, z)\in M\times M$ implies that $y-z=(x+y)-(x+z)\in Y$, $M\times M$ is contained in the proper subgroup $\{(x, y);\ x-y\in Y\}$ of $X\times X$. Next to prove the sufficiency, assume that $M\times M$ is contained in a proper closed subgroup Y^* of $X\times X$. We may assume that M generates X, for otherwise we have nothing to prove. $Y=\{x;\ (x, 0)\in Y^*\}$, where 0 is the identity element of X, is a proper closed subgroup of X. In fact, if $(x, 0)\in Y^*$ for every $x\in X$, then for $y\in X$, $(x, y)=(y, y)+(x-y, 0)\in Y^*$. This contradicts the fact that Y^* is a proper subgroup of $X\times X$. If $x, y\in M$, then $(y-x, 0)=(y, y)-(x, y)\in Y^*$,

$y-x\in Y$. Thus $M-x\subset Y$. This proves the lemma.

Theorem 6.2. *Let X be a locally compact second countable abelian group. Let P be a random measure on X, and λ be a probability measure on $\mathfrak{B}$. Suppose that* (i) *for every $x\in X$, $x+$supp$[\lambda]$ generates X, where* supp$[\lambda]$ *is the support of the measure λ,* (ii) *for every $\varphi\in C_0(X)$,*

$$\sup_{x\in X}\int_S (\varphi(x+\cdot),\mu)^2P(d\mu)<\infty ,$$

and (iii) *for every $\varphi\in C_0(X)$,*

$$\lim_{y\to x}\int_S [(\varphi(x+\cdot),\mu)-(\varphi(y+\cdot),\mu)]^2P(d\mu)=0 .$$

Then in order that the translation $\mathfrak{T}_\lambda(P)$ of P be identical with P, it is necessary and sufficient that $P(S^H)=1$.

Proof. The sufficiency part is proved without assumptions (i)-(iii). Since $P(S^H)=1$ implies that $P(S_\lambda)=1$ and for any $\varphi\in C_0(X)$,

$$P\{\mu;\,(\varphi,\mu)=(\varphi,\mu*\lambda)\}=1 , \tag{6.1}$$

we have by Theorem 4.1 that for $\varphi\in C_0^+(X)$,

$$l(\varphi;\mathfrak{T}_\lambda(P))=l\left(\int_X \varphi(x+\cdot)\lambda(dx);P\right)=\int_S e^{-(\varphi,\mu*\lambda)}P(d\mu)$$
$$=\int_S e^{-(\varphi,\mu)}P(d\mu)=l(\varphi;P) ,$$

It follows from Theorem 2.1 that $\mathfrak{T}_\lambda(P)=P$.

In order to prove the necessity, we define for each $\varphi\in C_0(X)$ a function $v(\cdot,\cdot;\varphi)$ on $X\times X$ by

$$v(x,y;\varphi)=\int_S (\varphi(x+\cdot),\mu)(\varphi(y+\cdot),\mu)P(d\mu) .$$

By (ii) and (iii), v is bounded and coutinuous on $X\times X$. Since $\mathfrak{T}_\lambda(P)=P$ implies (6.1) we have

$$v(x,y;\varphi)=\int_S (\varphi(x+\cdot),\mu*\lambda)(\varphi(y+\cdot),\mu*\lambda)P(d\mu) \tag{6.2}$$
$$=\int_{X\times X} v(x+x',y+y';\varphi)\lambda(dx')\lambda(dy') ,$$

By the assumption (i) and Lemma 6.1, the support of the product measure $\lambda\times\lambda$, which is identical with supp$[\lambda]\times$supp$[\lambda]$, generates $X\times X$. It follows from a famous theorem of *Choquet* and *Deny* [3] that $v(x,y;\varphi)$ is a constant function on $X\times X$. Hence we have

$$\int_S [(\varphi(x+\cdot),\mu)-(\varphi(y+\cdot),\mu)]^2 P(d\mu)$$
$$=v(x,x;\varphi)-2v(x,y;\varphi)+v(y,y;\varphi)=0 .$$

and therefore for every pair $x,y\in X$,

(6.3) $$P\{\mu;(\varphi(x+\cdot),\mu)=(\varphi(y+\cdot),\mu)\}=1 .$$

Let D be a countable dense subset of X. It follows from (6.3) that $P(S_0)=1$, where

$$S_0=\{\mu;(\alpha_i(x+\cdot),\mu)=(\alpha_i,\mu),i\geq 1,x\in D\}\in\mathfrak{A} .$$

If $\mu\in S_0$, then $(\varphi(x+\cdot),\mu)=(\varphi,\mu)$ for every $\varphi\in C_0(X)$ and $x\in X$, and therefore $\mu\in S^H$. Thus $S^H=S_0$ and $P(S^H)=1$.

The sufficiency part of the following theorem was essentially proved by *Doob* [6]. The necessity part may be regarded as a strengthening of results by *Dobrushin* [5] and *Goldman* [8].

Theorem 6.3. *Let X be a locally compact non-compact second countable abelian group. Let P be a point process on X, and let λ be a probability measure on $(X,\mathfrak{B})$. Suppose that the conditions* (i), (ii) *and* (iii) *of Theorem 6.2 are satisfied by P. Moreover we assume*

(6.4) $$\lim_{n\to\infty}\sup_{x\in X}\lambda^{n*}(A-x)=0 ,$$

for every bounded $A\in\mathfrak{B}$, where λ^{n*} is the n-fold convolution of λ with itself. Then in order that the random translation $\mathfrak{R}_\lambda(P)$ of P coincide with P, it is necessary and sufficient that P is a mixed Poisson process in the strict sense.

Proof. The sufficiency part follows from Theorem 5.4 and Theorem 6.2. Let us prove the necessity part. It follows from Theorem 5.3 that $\mathfrak{R}_\lambda(P)=P$ implies $P(S_{\lambda^{n*}})=1$, $\mathfrak{R}_{\lambda^{n*}}(P)=P$ for $n\geq 1$, and therefore $\lim_{n\to\infty}\mathfrak{R}_{\lambda^{n*}}(P)=P$. By (6.4) and Theorem 5.5 P must be a mixed Poisson process. Let $P=\mathfrak{M}(Q)$, where Q is a random measure. Then by Theorem 5.4 $Q(S_\lambda)=1$, $P=\mathfrak{R}_\lambda(P)=\mathfrak{M}(\mathfrak{T}_\lambda(Q))=\mathfrak{M}(Q)$. Hence by Theorem 4.2, $\mathfrak{T}_\lambda(Q)=Q$. By Theorem 4.3, Q satisfies the conditions (ii) and (iii) of Theorem 6.2. It follows from Theorem 6.2 that $Q(S^H)=1$. This proves the theorem.

Remark. If X is a non-compact closed subgroup of R^d, or more generally if there is a non-zero homomorphism from X to the additive group of real numbers, then (6.4) is implied by the assumption (i) of the theorem. However, the author does not know whether this is true for an arbitrary locally compact non-compact X.

Moreover it seems possible to replace the assumption (ii) and (iii) of Theorem 6.2 and Theorem 6.3 by some weaker ones. In fact, if X is the additive group of integers, then (ii) and (iii) may be replaced by

$$\sup_{x\in X}\int_S (\varphi(x+\cdot),\mu)P(d\mu)<\infty\ , \qquad \text{for} \quad \varphi\in C_0^+(X)\ .$$

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