HOROSPHERICAL REPRESENTATION OF LINEAR GROUPS

By

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§ 1. Introduction

In the study of matrix-groups, several types of triangular representations have been considered in the literature. For example, *McCoy*, [5], gave a characterisation of sets of matrices which can be simultaneously triangularised by a similarity transformation. Such sets are said to have Property p. For Lie-algebras and connected algebraic matrix groups, triangularisability is equivalent to solvability by the celebrated *Lie-Kolchin* Theorem [1], [4], [6].

Another type of triangularisation, called Property T, or sometimes called special triangular form, is the form in which the matrices appear as direct sum of monopotent triangular blocks. *Sinha*, [8], [9], gave characterisations of Property T for various sets and groups of matrices. This case turns out to be characteristic of nilpotency, just as Property P is for solvability.

In between these two extremes, there is a form of triangularisation, considered by *Gelfond* in [3]. In this triangularisation, the matrices appear with monopotent diagonal blocks with possible non-zero entries only on one side of these diagonal blocks. Following Gelfond, though in a slightly modified situation, we shall call such representation of matrices, a Horospherical Representation. We show below that every set of monopotent matrices having Property P, has such a representation. In particular, each matrix has such a representation unique upto similarity. Several necessary conditions are given for different types of matrix-sets, matrix-groups and algebraic matrix-groups to have horospherical representations. As a consequence we observe the following implication relation:

Property $T \Longrightarrow$ Horospherical Representation $\Longrightarrow$ Property P.

Finally we consider the special case of 2-dimensional vector-spaces and show that any set of linear transformations of such a space, has a horospherical representation if and only if it is commutative, so that in this case the concepts of Property T, of horospherical representation and of commutativity, all coincide.

We mention that though many of the results could be formulated and proved for slightly more general fields, we shall be limiting our considerations throughout to the field of complex-numbers.

§2. Definitions and Monopotent Sets

Unless otherwise specified we shall be considering $n\times n$ matrices acting as linear-transformations on an n-dimensional vector space V. Then we have the following:

Definitions:

1. An indecomposable set Ω of linear transformations of V, is said to have a Horospherical Representation if V has a descending sequence of Ω-admissible subspaces,

$$V=V_1\gneqq V_2\gneqq \cdots \gneqq V_{t+1}=0\ ,$$

such that V_i is a minimal Ω-admissible subspace of V_{i-1} with the property that the restriction of Ω to V_{i-1}/V_i, is a set of scalar matrices; $i=2,\cdots,t+1$.

We note that under these circumstances, with respect to a proper choice of basis for V, each element A of Ω, has the form:

$$A=\begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_t \end{bmatrix},$$

where the diagonal blocks A_i are scalar matrices representing the restriction of A to V_i/V_{i+1}; $i=1,\cdots,t$.

2. An arbitrary set Ω of linear transformations of V, is said to have a Horospherical-Representation if each set of its indecomposable components, have horospherical representations defined above.

We also recall the following standard definitions:

3. A pair of matrices $\{A, B\}$ is said to be 2-commutative in the additive sense, if $[[A, B], B]=0=[A, [A, B]]$, where $[X, Y]=XY-YX$. This merely implies that the additive commutator of $\{A, B\}$, commutes with both A and B.

4. Denoting $[A, B]$ by $A^{(1)}$ and defining $A^{(k)}=[A^{(k-1)}, B]$ inductively, we say that A is k-commutative with B if $A^{(k)}=0$.

5. If both A and B are mutually k-commutative then the pair $\{A, B\}$ is said to be k-commutative.

6. If each pair in a set, Ω, of matrices, is k-commutative then Ω is said to be k-commutative.

Apart from these we shall use rudimentary properties of Lie-algebras and Algebraic Linear Groups, for which we refer respectively to the standard works [4] and [1].

With these preliminaries we prove:

Theorem 1: *Let Ω be a set of monopotent triangularisable matrices. Then Ω has a horospherical representation.*

Proof: It clearly suffices to assume that Ω is indecomposable. Under our hypothesis, we can further assume that each $A\in\Omega$ has the form $A=A_s\cdot A_u$ where $A_s=\lambda(A)\cdot I$ is a scalar matrix with characteristic root $\lambda(A)$, while A_u is a unipotent triangular matrix, and $A_sA_u=A_u\cdot A_s$. {We recall that for non singular matrix x, we have what is called the unique Jordan Multiplicative Decomposition of the form $x=x_s\cdot x_u$, where x_s is semi-simple, x_u is unipotent, and $x_s\cdot x_u=x_u\cdot x_s$. Thus, over an algebraically closed field, x_s can be taken to be diagonal: [1]}.

With these observations, to prove the theorem, it clearly suffices to take Ω to be a unipotent set of triangular matrices.

Now let Σ be a finite subset of Ω. Then the algebra Ω^* generated by $\{A-I/A\in\Omega\}$, contains the subalgebra Σ^* generated by $\{\sigma-I|\sigma\in\Sigma\}$, which is clearly nilpotent by a well known Theorem of *Wedderburn*: [2], page 188.

Thus Ω^* is a locally nilpotent algebra of finite dimensional linear transformations, and hence is itself nilpotent. Let t be its index of nilpotency. Then $(\Omega^*)^t=0$ and t is minimal.

Now consider the Ω^*-admissible descending chain of subspaces:

$$V>V\Omega^*>V\Omega^{*2}>\cdots>V\Omega^{*t-1}>0\,,$$

where each containment is proper in view of the index of nilpotency of Ω^*.

Next it is easy to see that the lattice of Ω^*-admissible subspaces coincides with the Ω-admissible subspaces. Thus it suffices to prove that for each permissible i, $V\Omega^{*i}$ is minimal in $V\Omega^{*i-1}$ such that the restriction of Ω^* to $V\Omega^{*i-1}/V\Omega^{*i}$, is a zero-matrix. {Here we take $V\Omega^{*0}=V$}. Again $(V\Omega^{*i-1})\Omega^*=V\Omega^{*i}$, shows that the restriction is zero.

To prove minimality, let $W\subseteq V\Omega^{*i}$ such that (i) W is Ω^*-admissible, and (ii) Ω^* restricted to $V\Omega^{*i-1}/W$, is zero.

Then $(V\Omega^{*i-1})\cdot\Omega^*\subseteq W$ so that $V\Omega^{*i}\subseteq W$. Hence $W=V\Omega^{*i}$. This completes the proof. Q. E. D.

From the theory of Jordan-Canonical forms, we know that each $n\times n$ matrix is a direct sum of its primary components which can, in the canonical form, be taken to be monopotent triangular. Hence we have:

Cor. 1: *Every matrix has a horospherical representation, unique upto similarity.*

Since by a theorem of *Kolchin* [1], every multiplicatively closed set of unipotent matrices is triangularisable, we also have:

Cor. 2: *Any multiplicatively closed set of unipotent matrices has a horospherical representation.*

§ 3. Group Theoretic Conditions.

We apply the above Th. 1 to exhibit a non-trivial normal subgroup in every linear group, such that the subgroup has a horospherical representation.

Theorem 2: *Every finitely generated indecomposable linear group Ω, has a normal subgroup $H \neq I$, such that H has a horospherical representation.*

Proof: By virtue of Noether's Representation Theorem, there is a representation ρ of Ω such that for every $A \in \Omega$,

$$\rho(A) = \begin{bmatrix} \rho_1(A) & & * \\ & \ddots & \\ 0 & & \rho_t(A) \end{bmatrix},$$

where ρ_i are the irreducible constituents of ρ.

It is straight forward to verify that the map μ defined by,

$$\mu(A) = \begin{bmatrix} \rho_1(A) & & 0 \\ & \ddots & \\ 0 & & \rho_t(A) \end{bmatrix},$$

is a group homomorphism defined on Ω, and the kernel of μ is the normal subgroup H consisting of all $A \in \Omega$ such that

$$\rho(A) = \begin{bmatrix} I_1 & & * \\ & \ddots & \\ 0 & & I_t \end{bmatrix},$$

where I_i are identity matrices of the same dimension as the degree of the constituent ρ_i.

Applying Cor. 2 to Th. 1, H has a horospherical representation

Finally, since Ω is indecomposable, so clearly μ is not an isomorphism Thus $H \neq I$.

Cor. 1: *If Ω does not have a horospherical representation then H is proper.*

This corollary gives us an interesting simplicity criterion:

Cor. 2: *A finitely generated indecomposable linear group Ω is either not simple or has a horospherical representation.*

To apply Th. 1 to certain nilpotent linear groups, we recall that in [7], the following was proved:

Lemma 3: *If $C=(A, B)=ABA^{-1}B^{-1}$ is unipotent and commutes with both the invertible linear transformations A and B of a vector space V, then the B primary components of V, are A-invariant.*

It follows that the following holds:

Lemma 4: *If $\mathscr{L}=\{A_1, \cdots, A_m\}$ is a finite set of linear transformations of V, such that for each i, j, (A_i, A_j) is unipotent and commutes with both A_i and A_j, then $V=V_1\oplus\cdots\oplus V_t$ such that for each i and j, A_i restricted to V_j is monopotent.*

We use these results to prove:

Theorem 5: *Let Ω be a finitely generated nilpotent linear group with the generators $\{A_1, A_2, \cdots, A_m\}$, such that for each i, j, (A_i, A_j) is unipotent and commutes with each A_k. Then Ω has a horospherical representation.*

Proof: By Lemma 4, the restriction of each A_i to any Ω-indecomposable component of V_1 is monopotent. Further, since (A_i, A_j) is unipotent and commutes with each A_k, so $(A_i, A_j)-I$ lies in the radical of the associative algebra generated by $\{A_1, A_2, \cdots, A_m\}$. Then, by a trivial modification of McCoy's Criterion for triangularisability, we conclude that $\{A_1, \cdots, A_m\}$, and hence their restrictions to the Ω-indecomposable components of V, form a triangularisable set of matrices. Now it follows as a straight consequence of Th. 1 above, that Ω has a horospherical representation. Q. E. D.

In order to obtain conditions for horospherical representations of algebraic linear groups, we recall the following results from [1] Th. 11.1 and [6] p. 30 respectively:

Lemma 6: *If Ω is a connected nilpotent linear group, then Ω_s, the set of semi-simple parts of the elements of Ω, belong to the centre of Ω.*

Lemma 7: *A connected algebraic linear group is solvable if and only if it is triangularizable.*

We apply these to obtain:

Theorem 8: *If Ω is a connected nilpotent algebraic linear group, then Ω has a horospherical representation. (Conversely) If Ω is a linear group such that Ω_s is contained in its centre, and Ω has a horospherical representation then Ω is nilpotent.*

Proof: Let Ω be a connected nilpotent algebraic linear group. By Lemma 6, Ω_s is central in Ω. We can also assume Ω to be indecomposable as before. Then by Th. 1 in [7], Ω_s is a monopotent set of matrices. Then in view of the Jordan Multiplicative Decomposition $A=A_s\cdot A_u$, where A_s is semi-simple and A_u is unipotent, we deduce that Ω is itself a monopotent set of matrices. Further, by Lemma 7, Ω is triangularisable. Hence, by Th. 1, Ω has a horospherical representation.

For the second part of the theorem, we again make the simplification of taking Ω to be indecomposable. Then, as Ω_s is central by hypothesis, the same argument as above gives that Ω is a monopotent set of matrices. The rest of the argument is as in the proof of Th. 1 of [9]. By virtue of horosperical representation, we can assume that each $A\in\Omega$ has the form,

$$A=\begin{bmatrix} A_1 & & & * \\ & A_2 & & \\ & & \ddots & \\ 0 & & & A_t \end{bmatrix},$$

where $A_i=\lambda_A\cdot I_i$, where λ_A is the unique characteristic root of A, and I_i is the unit matrix of suitable dimensions. Since each A in Ω is nonsingular, so $A=(\lambda_A\cdot I)\cdot(\lambda_A^{-1}\cdot A)=A_s\cdot A_u$ where $A_s=\lambda_A\cdot I$ and $A_u=\lambda_A^{-1}\cdot A$.

Now, clearly, the set $\{\lambda_A|A\in\Omega\}$ is a multiplicative subgroup N of the complex field, and hence N is abelian also. Further $\mathscr{U}=\{A_u|A\in\Omega\}$ is a triangularisable unipotent group, and hence nilpotent: [1].

Since under our hypotheses, N and $\mathscr{U}$ commute elementwise and $N\cap\mathscr{U}=I$, so $\Omega=N\times\mathscr{U}$, whence Ω is nilpotent. Q.E.D.

In [9], the equivalence of Property T and nilpotence of connected algebraic linear groups, was established. Hence we have:

Cor. 1: *For connected algebraic linear groups, Property T implies horospherical representation.*

In the next section we shall establish the same result for arbitrary sets of linear transformations.

§4. Additive Commutator Conditions.

We prove first:

Theorem 9: *Let Ω be a finite set of 2-commutative linear transformations. Then the enveloping Lie algebra $\bar{\Omega}$ generated by Ω, has a horospherical representation.*

Proof: Let $\Omega=\{A_1, \cdots, A_m\}$. Since Ω is 2-commutative, so the additive commutators $[A_i, A_j]$ belongs to the centre of Ω.

Now $\bar{\Omega}$ is generated, as a vector space, by the Lie-monomials $[\cdots[A_{i_1}, A_{i_2}], \cdots, A_{i_s}]$, which are all zero if $s>2$ in view of the above comment. Thus $\bar{\Omega}$ has a finite vector space basis consisting of the elements $\{A_1, \cdots, A_m, [A_i, A_j]\cdots\}$. Hence $[\bar{\Omega}, \bar{\Omega}]$, the commutator ideal, is contained in the centre of $\bar{\Omega}$. Then if X is any element of $\bar{\Omega}$, and $\operatorname{ad} X$ is the adjoint of X, we have that for any vector $u \in \bar{\Omega}$,

$$u(\operatorname{ad} x)^s=[\cdots[u, x], \cdots, x]=0 \quad \text{for} \quad s>1 .$$

Thus each $\operatorname{ad} X$ is nilpotent so that $\bar{\Omega}$ is a nilpotent Lie-algebra, and it is well known that for such a Lie-algebra, of linear transformations of a finite dimensional vector space V, we can decompose V into a direct sum of indecomposable Ω-admissible subspaces:

$$V=V_1\oplus \cdots \oplus V_t ,$$

such that the restriction of the elements of $\bar{\Omega}$ to any V_i, are monopotent and triangularisable: [4].

Then by Th. 1, each of these indecomposable components of $\bar{\Omega}$, has a horospherical representation, and hence so has $\bar{\Omega}$. Q.E.D.

Cor. 1: *Every commutative set of linear transformations has a horospherical representation.*

Finally, we have:

Theorem 10: *If Ω is a set of linear transformations of the vector space V such that,*

(i) *Ω has Property P, and*

(ii) *Ω is k-commutative in the additive sense for some finite k, then Ω has a horospherical representation.*

Proof: From [4], page 40, k-commutativity implies that V has a direct sum decomposition into Ω-admissible indecomposable subspace:

$$V=V_1\oplus\cdots\oplus V_t ,$$

such that the set of restrictions of the elements of Ω to any of the V_i, is a monopotent set. This fact, combined with hypothesis (i) of the theorem and Th. 1, gives a horospherical representation for Ω. Q.E.D.

Now we remark that in [8], it has been shown that hypotheses (i) and (ii) of Th. 10 above, are equivalent to Property T for Ω. Thus we conclude:

Cor. 1: *If a set Ω of linear transformations has Property T, then it has a horospherical representation.*

§ 5. Special Case of dim. 2.

For this section, we assume that dim. $V=2$. We then have:

Theorem 11: *A set Ω of linear transformations of V, has a horospherical representation if and only if Ω is commutative.*

Proof: If Ω is commutative then by Cor. to Th. 9 above, Ω has a horospherical representation.

Conversely, if Ω has a horospherical representation, then, since, dim. $V=2$, so either Ω consists of only scalar matrices or V has a 1-dimensional Ω-invariant subspace $V_1=\{v_1\}$, where v_1 is the basis of V_1. Hence for all $A\in\Omega$, $Av_1=\lambda(A)v_1$, and A has the form

$$A=\begin{bmatrix} \lambda(A) & \nu(A) \\ 0 & \mu(A) \end{bmatrix}.$$

If $\nu(A)=0$ for every A, then $V=V_1\oplus V_2$ where V_2 is also Ω-invariant, so that Ω is diagonable and hence abelian.

On the other hand, if V_1 is the unique 1-dimensional Ω-invariant subspace of V, then let $\{v_1, v_j\}$ be a basis for V^*, the dual vector space of V. Then $(v_i, v_j^*)=\delta_{ij}$, the Kronecker-δ, where the parenthesis denotes the usual inner product of vector spaces.

Now v_2^* is incident of v_1, and $V_1=\langle v_1\rangle$ is Ω-invariant, so $V_2^*=\langle v_2^*\rangle$ is also Ω^*-invariant where Ω^* is the set of transposes of the elements of Ω. Again V_2^* must be the unique 1-dimensional Ω^*-invariant subspace of V^* or else Ω^* and hence Ω, will be diagonal.

Now let $A\in\Omega$. Then,

$$Av_1=\lambda(A)\cdot v_1\ ,$$
$$Av_2=\nu(A)v_1+\mu(A)v_2\ ,$$

so that

$$v_1^*A^*=\mu(A)v_1^*+\nu(A)v_2^*\ ,$$
$$v_2^*A^*=\lambda(A)v_2^*\ .$$

Hence

$$\begin{aligned}(Av_2, v_2^*)&=\nu(A)\cdot(v_1, v_2^*)+\mu(A)(v_2, v_2^*)\ ,\\ &=\mu(A)\ ,\\ &=(v_2, v_2^*A^*)\ ,\\ &=\lambda(A)(v_2, v_2^*)\ ,\\ &=\lambda(A)\ .\end{aligned}$$

Thus we have proved that Ω consists of monopotent triangular matrices only. Then it is easy to verify that such a set is commutative.

This completes the proof. Q. E. D.

Cor 1: *If dim. $V=2$, then for any set Ω of linear transformations of V, the notions of Pr operty T, horospherical representation and commutativity all coincide.*

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