APPROXIMATION OF AN ENTIRE FUNCTION

By

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1. Let $f(x)$ be a real valued continuous function defined on $[-1, 1]$ as usual let

$$E_n(f) \equiv \inf_{p \in \pi_n} \|f-p\| \qquad \text{for} \quad n=0, 1, 2, \cdots \; p \in \pi_n .$$

Where the norm is the maximum norm on $[-1, 1]$ and π_n denotes the set of all polynomials with real coefficients of degree at most n. *Bernstein* ([1], p. 118) proved that

$$\lim_{n\to\infty} E_n^{1/n}(f) = 0 , \tag{1.1}$$

holds if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function. Further, let $f(z)$ be an entire function and

$$M(r) = \max_{|z|=r} |f(z)| ,$$

$$\begin{cases} \overline{\underline{\lim}}_{r\to\infty} \dfrac{\log\log M(r)}{\log r} = \begin{matrix} \rho \\ \zeta \end{matrix} & (0 \leq \zeta \leq \rho \leq \infty) \\ \overline{\underline{\lim}}_{r\to\infty} \dfrac{\log M(r)}{r^{\rho}} = \begin{matrix} T \\ t \end{matrix} & \begin{matrix} (0<\rho<\infty) \\ (0 \leq t \leq T \leq \infty) \end{matrix} \end{cases} \tag{1.2}$$

where ρ, ζ and T, t denote the order and lower order type and lower type respectively of an entire function $f(z)$.

Further, *Bernstein* ([1], p. 114) has shown that there exists constant ρ (positive) α, T (non-negative) such that

$$\limsup_{n\to\infty} n^{1/\rho} E_n^{1/n}(f) = \alpha , \tag{1.3}$$

if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function of order ρ and type T.

Recently *Varge* ([4], Theorem 1) has shown that

1.4) $$\limsup_{n\to\infty} \left\{ \frac{n \log n}{\log \dfrac{1}{E_n(f)}} \right\} = \rho ,$$

where ρ is the non-negative real number if and only if $f(x)$ is the restriction to $[-1, 1]$ of an entire function $f(z)$ of order ρ.

In this note, the above result of *Varge* has been extended to lower order of an entire function $f(z)$.

2. Theorem. *Let $f(x)$ be a real-valued continuous function on $[-1, 1]$, then*

$$\text{(2.1)} \qquad \operatorname*{Min}_{\{n_h\}} \limsup_{h\to\infty} \left\{ \frac{\log \dfrac{1}{E_{n_h}(f)}}{n_h \log n_{h-1}} \right\} = \frac{1}{\zeta} .$$

Where ζ is a non-negative real number, if $f(x)$ is the restriction to $[-1, 1]$ of an entire function $f(z)$ of lower order ζ. $\{n_h\}$ is the subsequence of non-negative integers such that $n_0 < n_1 < n_2 < \cdots$.

We need the following lemma for our purpose.

Lemma ([3], Theorem 2). *Let $f(z) = \sum_0^\infty a_n z^n$ be an entire function of lower order ζ then*

$$\text{(2.2)} \qquad \frac{1}{\zeta} = \min_{\{n_h\}} \limsup_{h\to\infty} \left\{ \frac{\log \dfrac{1}{|a_{n_h}|}}{n_h \log n_{h-1}} \right\} .$$

Where $\{n_h\}$ is the subsequence of non-negative integers such that $n_0 < n_1 < n_2 < \cdots$.

Proof of Theorem. First, assume that $f(x)$ has an analytic extension $f(z)$ which is an entire function of lower order ζ, where $0 \le \zeta \le \infty$. Following *Bernstein* original proof we have ([2], p. 78) for each $n \ge 0$

$$\text{(2.3)} \qquad E_n(f) \le \frac{2B(\sigma)}{\sigma^n(\sigma-1)} . \qquad \text{for} \quad \sigma > 1$$

Where $B(\sigma)$ is the maximum of the absolute value of $f(z)$ on E_σ, with $\sigma > 1$ denotes the closed interior of ellipse with foci at ± 1 major semi- axis $\left(\frac{\sigma^2+1}{2\sigma}\right)$ and mimor semi-axis $\left(\frac{\sigma^2-1}{2\sigma}\right)$. The closed discs $J_1(\sigma)$ and $J_2(\sigma)$ bound the ellipse E_σ in the sense that

$$J_1(\sigma) \equiv \left\{ z \,\middle|\, |z| \le \frac{\sigma^2-1}{2\sigma} \right\} \subset E_\sigma \subset J_2(\sigma) \equiv \left\{ z \,\middle|\, |z| \le \frac{\sigma^2+1}{2\sigma} \right\} .$$

From this inclusion, it follows by definition that

$$\text{(2.4)} \qquad M_f\left(\frac{\sigma^2-1}{2\sigma}\right) \le B(\sigma) \le M_f\left(\frac{\sigma^2+1}{2\sigma}\right) , \qquad \text{for all} \quad \sigma > 1$$

From (2.4) we can verify that

$$\begin{aligned} \frac{\rho}{\zeta} &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log M(\sigma)}{\log \sigma} \\ &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log B(\sigma)}{\log \sigma} . \end{aligned} \tag{2.5}$$

Consequently from (2.3), we have

$$E_n(f) \leq \frac{CB(\sigma)}{\sigma^n} , \tag{2.6}$$

where C is some constant since $\sigma>1$, $\frac{2}{\sigma-1}\leq C$. From (2.6), we can have for some $\eta>0$

$$\begin{aligned} \sum_{k=0}^{\infty} E_k(f)\sigma^k &\leq \sum_{k=0}^{\infty} \frac{CB(\sigma+\eta)}{(\sigma+\eta)^k}\cdot\sigma^k \\ &= CB(\sigma+\eta)\sum_{k=0}^{\infty}\left(\frac{C}{\sigma+\eta}\right)^k \\ &\leq \frac{C'B(\sigma+\eta)(\sigma+\eta)}{\eta} . \end{aligned} \tag{2.7}$$

Now we have from ([4] p. 12),

$$B(\sigma)\leq|P_0(z)|+2\sigma\sum_{k=0}^{\infty} E_k\sigma^k , \tag{2.8}$$

where $|P_0(z)|$ is a constant. E_k is a decreasing sequence of real number along k.

Let us write

$$H(\sigma)=\sum_{k=0}^{\infty} E_k\sigma^k , \tag{2.9}$$

which represents an entire function, then we have from (2.7) and (2.8).

$$B(\sigma)\leq C'\sigma H(\sigma)\leq C''\sigma(\sigma+\eta)B(\sigma+\eta) , \tag{2.10}$$

where C', C'' are constants. From (2.5) and (2.10) we can verify that

$$\begin{aligned} \frac{\rho}{\zeta} &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log B(\sigma)}{\log \sigma} \\ &= \lim_{\sigma\to\infty} \begin{matrix}\sup\\ \inf\end{matrix} \frac{\log\log H(\sigma)}{\log \sigma} . \end{aligned} \tag{2.11}$$

Now applying lemma to $H(\sigma)$, we have the required result

$$\frac{1}{\zeta} = \operatorname*{Min}_{\{n_h\}} \limsup_{h\to\infty} \left\{ \frac{\log \dfrac{1}{E_{n_h}(f)}}{n_h \log n_{h-1}} \right\}.$$

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