A NOTE ON COMMON FIXED POINTS

By

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1. Introduction:

Throughout this paper we will write X for a complete metric space (X, d). The well known Banach contraction principle states that if

$$d(Tx, Ty) \leqslant Rd(x, y) \ ,$$

for all $x, y \in X$ and $0 < R < 1$, where T is an operator mapping X into itself then T has a unique fixed point.

For two operators T_1 and T_2 each mapping X into itself Kannan [1] investigated a sufficient condition for the existence of a common and unique fixed point in X. He has proved the following theorem.

Theorem. A [1, Theorem 1]
If T_1 and T_2 are two operators each mapping X into itself and if

$$d(T_1(x), T_2(y)) \leqslant \alpha[d(x, T_1(x)) + d(y, T_2(y))] \ ,$$

where $x, y \in X$ and $0 < \alpha < \frac{1}{2}$, then T_1 and T_2 have a unique common fixed point.

If T_1 is identical with T_2 in the above theorem, then we have

Theorem. B [1, Theorem 2]
If T be an operator mapping X into itself and if

$$d(T(x), T(y)) \leqslant \alpha[d(x, T(x)) + d(y, T(y))] \ ,$$

where $x, y \in X$ and $0 < \alpha < \frac{1}{2}$, then T has a unique fixed point in X.

More recently *Singh* [2] obtained the following generalization of Theorem B.

Theorem. C [2, Theorem 1]
If T be an operator mapping X into itself and if T^n (n is a positive integer) satisfies

$$d(T^n(x), T^n(y)) \leqslant \alpha[d(x, T^n(x)) + d(y, T^n(y))] \ ,$$

for all $x, y \in X$ *and* $0<\alpha<\frac{1}{2}$, *then* T *has a unique fixed point in* X.

In the present note we establish a more general theorem in this direction. Theorem A, B and C become corollaries to our result.

2.1 We prove the following theorem. We mention also a few additional consequences of the main theorem.

Theorem. 1

If T_1 *and* T_2 *are two operators mapping* X *into itself such that*

$$d(T_1^p(x), T_2^q(y)) \leqslant \alpha d(x, T_1^p(x)) + \beta d(y, T_2^q(y)) ,$$

where $x, y \in X$, p *and* q *are positive integers,* $\alpha>0$, $\beta>0$, $\alpha+\beta<1$, *then* T_1 *and* T_2 *have unique and common fixed point.*

2.2 Proof.

We define a sequence of elements $\{x_n\}$ in X as follows. Let $x \in X$ be arbitrary. Set

$$x_1=T_1^p(x), \quad x_2=T_2^q(x_1), \quad x_3=T_1^p(x_2), \quad x_4=T_2^q(x_3) \quad \text{and so on.}$$

In general $x_{2n+1}=T_1^p(x_{2n})$ and $x_{2(n+1)}=T_2^q(x_{2n+1})$.

Now
$$\begin{aligned} d(x_1, x_2) &= d(T_1^p(x), T_2^q(x_1)) \leqslant \alpha d(x, T_1^p(x)) + \beta d(x_1, T_2^q(x_2)) \\ &= \alpha d(x, x_1) + \beta d(x_1, x_2) . \end{aligned}$$

Therefore
$$d(x_1, x_2) \leqslant \frac{\alpha}{1-\beta} d(x, x_1) .$$

Also
$$\begin{aligned} d(x_2, x_3) &= d(T_2^q(x_1), T_1^p(x_2)) \\ &\leqslant \alpha(x_2, T_1^p(x_2)) + \beta d(x_1, T_2^q(x_1)) , \end{aligned}$$

and hence
$$\begin{aligned} d(x_2, x_3) &\leqslant \frac{\beta}{1-\alpha} d(x_1, x_2) \\ &\leqslant \frac{\beta}{1-\alpha} \cdot \frac{\alpha}{1-\beta} d(x, x_1) , \end{aligned}$$

Put $r_1=\frac{\alpha}{1-\beta}$, $r_2=\frac{\beta}{1-\alpha}$. Since $\alpha+\beta<1$, r_1 and $r_2<1$. Then

$$d(x_{2n}, x_{2n+1}) \leqslant r_2 r_1 \cdots r_1 r_2 r_1 d(x, x_1) = r_2^n r_1^n d(x, x_1) ,$$

$$d(x_{2n+1}, x_{2(n+1)}) \leqslant r_1^{n+1} r_2^n d(x, x_1) .$$

Hence for $m=2\nu$, we have

$$\begin{aligned}
d(x_m, x_{m+n}) &\leqslant d(x_m, x_{m+1})+d(x_{m+1}, x_{m+2})+\cdots+d(x_{m+n-1}, x_{m+n})\\
&\leqslant (r_2^\nu r_1^\nu+r_2^\nu r_1^{\nu+1}+r_2^{\nu+1}r_1^{\nu+1}+\cdots+\cdots n \text{ terms}) \times d(x, x_1)\\
&< (r_2^\nu r_1^\nu+r_2^\nu r_1^{\nu+1}+r_2^{\nu+1}r_1^{\nu+1}+\cdots+\cdots \text{up to infinity}) \times d(x, x_1)\\
&= [r_2^\nu r_1^\nu(1+r_1r_2+r_1^2r_2^2+\cdots)+r_1^{\nu+1}r_2^\nu(1+r_1r_2+r_1^2r_2^2+\cdots)]\times d(x, x_1)\\
&= r_2^\nu r_1^\nu (1+r_1)\frac{1}{1-r_1r_2}\times d(x, x_1)\\
&\to 0 \text{ as } \nu\to\infty \text{ i.e. } m\to\infty .
\end{aligned}$$

Similarly, for $m=2\nu+1$

$$\begin{aligned}
d(x_m, x_{m+n}) &\leqslant (r_2^\nu r_1^{\nu+1}+r_1^{\nu+1}r_2^{\nu+1}+r_1^{\nu+2}r_2^{\nu+1}+r_1^{\nu+2}r_2^{\nu+2}+\cdots)\times d(x, x_1)\\
&= \{r_2^\nu r_1^{\nu+1}(1+r_1r_2+r_1^2r_2^2+\cdots)+r_1^{\nu+1}r_2^{\nu+1}(1+r_1r_2+r_1^2r_2^2+\cdots)\times d(x, x_1)\\
&= r_1^{\nu+1}r_2^\nu(1+r_2)\frac{1}{1-r_1r_2}\times d(x, x_1)\\
&\to 0 \text{ as } \nu\to\infty \text{ i.e. } m\to\infty .
\end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence. Since the space is complete there exists $x_0\in X$ such that

$$\lim_{n\to\infty} x_n=x_0 .$$

We first show that

$$T_1^p(x_0)=T_2^q(x_0)=x_0 .$$

We have

$$d(x_0, T_1^p(x_0))\leqslant d(x_0, x_t)+d(x_t, T_1^p(x_0))=d(x_0, x_t)+d(T_2^q(x_{t-1}), T_1^p(x_0)) .$$

Where t is taken to be even.
Hence

$$d(x_0, T_1^p(x_0))<d(x_0, x_t)+\alpha d(x_0, T_1^p(x_0))+\beta d(x_{t-1}, T_2^q(x_{t-1})) ,$$

or, $$(1-\alpha)d(x_0, T_1^p(x_0))<d(x_0, x_t)+\beta d(x_{t-1}, x_t) .$$

The expression on the right hand side can be made arbitrarily small by choosing t sufficiently large. Therefore,

$$d(x_0, T_1^p(x_0))=0 .$$

That is to say that $T_1^p(x_0)=x_0$. Similarly $x_0=T_2^q(x_0)$.

We further show that x_0 is the unique common fixed point of T_1^p and T_2^q. Suppose y_0 also satisfies

$$T_1^p(y_0)=T_2^q(y_0)=y_0 .$$

Then

$$d(x_0, y_0)=d(T_1^p(x_0), T_2^q(y_0))\leqslant \alpha d(x_0, T_1^p(x_0))+\beta d(y_0, T_2^q(y_0))=0 ,$$

and

$$x_0=y_0 .$$

Finally we obtain that x_0 is the unique common fixed point of T_1 and T_2. For,

$$T_1^p(x_0)=x_0 \Longrightarrow T_1^p(T_1(x_0))=T_1(T_1^p(x_0))=T_1(x_0)$$

$$\Longrightarrow T_1(x_0)=x_0, \text{ since } T_1^p \text{ has a unique fixed point } x_0.$$

Similarly $T_2(x_0)=x_0$.

Moreover, x_0 is the only fixed point of T_1 and of T_2. Suppose if possible $z_0\neq x_0$ and $T_1(z_0)=T_2(z_0)=z_0$. Then

$$\begin{aligned} d(x_0, z_0)&=d(T_2(x_0), T_1(z_0))=d(T_2^q(x_0), T_1^p(z_0)) \\ &\leqslant \alpha d(z_0, T_1^p(z_0))+\beta d(x_0, T_2^q(x_0))=0 . \end{aligned}$$

Which implies $x_0=z_0$.

This completes the proof of the theorem.

2.3 Remarks.

i) For $q=p=1$, and $\alpha=\beta$ we get theorem A.

ii) For $p=q=1$, $T_1=T_2$ and $\alpha=\beta$ we get theorem B.

iii) For $p=q$, $T_1=T_2$ and $\alpha=\beta$ we get theorem C.

2.4 As simple consequences we state the following theorems.

Theorem. 2

Let T be an operator mapping X into itself such that

$$(*) \qquad d(T^p(x), T^q(y))\leqslant \alpha d(x, T^p(x))+\beta d(y, T^q(y)) .$$

where $x, y\in X$, p and q are positive integers, $\alpha>0$, $\beta>0$, and $\alpha+\beta<1$, then T has a unique fixed point in X.

Theorem. 3

Let T_1 be an operator mapping X into itself such that T_1 satisfies $(*)$ and if T_2 be an operator mapping X into itself such that $T_1T_2=T_2T_1$, then T_1 and T_2 have a unique common fixed point.

Theorem 2 is obtained by putting $T_1=T_2$ in Theorem 1.

For the proof of theorem 3 it is sufficient to note that if x_0 is the unique fixed point of T_1, then $T_1(x_0)=x_0$ implies $T_1T_2(x_0)=T_2T_1(x_0)=T_2(x_0)$ which implies $T_2(x_0)=x_0$, that is to say that x_0 is a fixed point of T_2 also.

REFERENCES

[1] Kannan, R. *Some Results on Fixed Points.* Bull. Cal. Math. Soc. Vol. 60 No. 182 pp. 71-76.

[2] Singh, S. P. *Some Results on Fixed Points.* Yokohama Math. Journal. Vol. XVII, No. 2 pp. 61-64.

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