A COUNTEREXAMPLE TO A PROOF OF HOMMA*

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1. Introduction. In this note we describe a counterexample to the proof of Theorem 1 of [1], which is essentially the same as Lemma 2.2 of [2]. The following is a simplified version of this Theorem which will suffice for our purposes:

Let E^n be euclidean n-space. If $f: P \to Q$ is a piecewise linear mapping of a polyhedron P onto a polyhedron Q, and $g: P \to E^n$ is a piecewise linear mapping satisfying

$$n > \dim Q + 2 \operatorname*{Max}_{q \in Q} \dim f^{-1}(q) ,$$

then for any $\epsilon > 0$ there is a piecewise linear mapping $h: P \to E^n$ satisfying

(1) $d(h, g) = \sup_{p \in P} d(h(p), g(p)) < \epsilon$,

(2) *h is non-degenerate*

(α_1) $h | f^{-1}(q)$ *is a homeomorphism*

(α_2) $hf^{-1}(q_1) \cap hf^{-1}(q_2) =$ *one point or* ϕ, *for* $q_1 \neq q_2 \in Q$.

In the proof of Theorem 1 in [1], once subdivisions of P and Q are obtained so that f and g are simplicial with respect to these subdivisions, no more subdividing is done. The method of proof is essentially general positioning $g(P)$ in E^n.

2. Definitions. We use the standard terminology of piecewise linear topology following [3].

If $f: K \to L$ is a simplicial mapping from a complex K onto a complex L, and $x \in |L|$, then $f^{-1}(x)$ is said to be *the fibre over x*. Note, that if for each $x \in |L|$, $\dim f^{-1}(x) \leq 1$, and if $\Delta \in K$, with Δ' a 1-dimensional face of Δ, such that $f(\Delta')$ $=$point, and $\dim(f^{-1}(y) \cap \Delta) = 1$, then $f^{-1}(y) \cap \Delta$ is parallel to Δ'.

If K is a complex and $f: |K| \to E^n$ is a continuous mapping of $|K|$ into E^n such that for any $\sigma \in K$, $f | \sigma : \sigma \to E^n$ is linear, then f is called a *semi-simplicial*

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mapping of K into E^n.

We denote the join of simplices σ and τ by $\sigma*\tau$.

3. Example. (α_2) states that if the images of two fibres meet, they meet in a point. The important consideration is that they meet in a spine of one of them. The following example shows that the images of two connected fibres may always be forced to meet in precisely two points unless P is further subdivided.

We shall construct a simplex K containing three principal simplices σ, σ' and τ; and consider a semi-simplicial mapping $g: K \to E^n$. Let $\dim \tau = n-2$, $\dim \sigma = \dim \sigma' \leq (n/2)-1$, and assume $\dim \tau + \dim \sigma - n > 0$. Let v_σ, w_σ be vertices of σ and $v_{\sigma'}, w_{\sigma'}$ be vertices of σ', where $\sigma = v_\sigma * \sigma_1$ and $\sigma' = v_{\sigma'} * \sigma_1'$. Let v_1 and v_2 be two vertices of τ. We form K by identifying v_σ with v_1, $v_{\sigma'}$ with v_2, and σ_1 with σ_1' where w_σ is identified with $w_{\sigma'}$; and obtain a connected complex. Now we have

$$\sigma \cap \tau = v_\sigma, \quad \sigma' \cap \tau = v_{\sigma'}, \quad \sigma' \cap \sigma = \sigma_1 .$$

Let $\tilde{\tau}$=face of τ opposite v_σ and $\tilde{\sigma}'$=face of σ' opposite w_σ. Let $L = \tilde{\sigma}' \cup \tilde{\tau}$. Let $f: |K| \to L$ be a simplicial mapping satisfying: $f \,|\, |L|$ is the identity, $f(v_\sigma) = v_\sigma'$, $f(w_\sigma) = v_\sigma'$. Then f is defined over all of $|K|$.

Let $P = |K|$ and $Q = |L|$. Note that

$$\dim Q + 2 \max_{q \in Q} \dim f^{-1}(q) = (n-3) + 2(1) = n-1 < n .$$

Let $g: |K| \to E^n$ be a semi-simplicial mapping in general position. We can have g such that

$$g(\tau) \cap g(\sigma) - g(\tau \cap \sigma) \neq \phi ,$$

and

$$g(\tau) \cap g(\sigma') - g(\tau \cap \sigma') \neq \phi .$$

Now there is an $\epsilon > 0$ such that given any semi-simplicial mapping $h: |K| \to E^n$, if $\sup d(h(x), g(x)) < \epsilon$ then

$$h(\tau) \cap h(\sigma) - h(\tau \cap \sigma) \neq \phi ,$$

and

$$h(\tau) \cap h(\sigma') - h(\tau \cap \sigma') \neq \phi .$$

We show that (α_2) cannot be guaranteed to be satisfied for any such h.

Let $x \in h(\sigma^\circ) \cap h(\tau^\circ)$, now the fibre $F_{x,\sigma}$, in $h(\sigma)$ which passes through x is parallel to $\langle h(v_\sigma), h(w_\sigma) \rangle$; and $F_{x,\tau}$ is parallel to $\langle h(v_{\sigma'}), h(v_\sigma) \rangle$. There is a

$y \in h(\sigma_1)$ such that $y \in F_{x,\sigma}$. We have $F_{y,\sigma'}$ parallel to $<h(v_{\sigma'}), h(w_\sigma)>$. We claim that $F_{y,\sigma'}$ lies in the 2-plane formed by $F_{x,\sigma}$ and $F_{x,\tau}$. This is true since these three line segments all lie in 2-planes parallel to the 2-plane determined by $<h(w_\sigma), h(v_\sigma), h(v_{\sigma'})>$; but $F_{x,\sigma}$ and $F_{x,\tau}$ lie in the same 2-plane since they meet at x; and $F_{x,\sigma}$ and $F_{y,\sigma'}$ lie in the same 2-plane since they meet at y; therefore $F_{x,\tau}$ and $F_{y,\sigma'}$ lie in the same 2-plane. The line through $F_{x,\tau}$ intersects the line through $F_{y,\sigma'}$; so that by general positioning the vertices of σ, σ' and τ we cannot guarantee that $F_{x,\tau} \cap F_{y,\sigma'} = \phi$. If $F_{x,\tau} \cap F_{y,\sigma'} \neq \phi$, let $x_1 \in \sigma$, $x_2 \in \tau$ such that, $h(x_1) = h(x_2) = x$, then $hf^{-1}f(x_1) = F_{x,\sigma} \cup F_{y,\sigma'}$, $hf^{-1}f(x_2) = F_{x,\tau}$ and $(F_{x,\sigma} \cup F_{y,\sigma'}) \cap F_{x,\tau} = 2$ points, not satisfying (α_2).

Remarks. The reason why *Homma's* Lemma 1 and Theorem 2 of [1] cannot be used to avoid this difficulty is that Homma requires the moving of the vertices of σ while keeping the vertices of $\sigma' \cup \tau$ fixed. But this cannot be done when the vertices of σ are contained in $\sigma' \cup \tau$. In general, as long as some of the vertices of σ are contained in $\sigma' \cup \tau$, difficulties may arise.

REFERENCES

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