RECURRENT FINSLER SPACE OF SECOND ORDER

By

B. B. SINHA and S. P. SINGH

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Summary. The recurrent *Finsler* spaces have been studied by *Mishra* and *Pandey* [1][1], *Sen* [2] and *Sinha* and *Singh* [3]. *Chaki* and *Roy Chowdhary* [4] have introduced the Ricci recurrent spaces of second order in the classical Riemannian geometry. The purpose of present paper is to define the recurrent Finsler spaces of second order and to study the properties of the recurrence vector and tensor fields and the curvature tensor fields in this space. The notations of *Rund* [5] have been followed in the sequel.

1. Introduction. We consider an n-dimensional Finsler space F_n in which the Berwald's curvature tensor fields are given by

$$H^i_{jk}=\partial_k\dot{\partial}_j G^i-\partial_j\dot{\partial}_k G^i+G^i_{kr}\dot{\partial}_j G^r-G^i_{rj}\dot{\partial}_k G^r \tag{1.1}$$

and

$$H^i_{jkh}=\partial_h G^i_{jk}-\partial_k G^i_{jh}+G^r_{jk}G^i_{rh}-G^r_{jh}G^i_{rk}+G^i_{rjh}\dot{\partial}_k G^r-G^i_{rjk}\dot{\partial}_h G^r\ , \tag{1.2}$$

where $\quad \partial_h=\dfrac{\partial}{\partial x^h} \quad$ and $\quad \dot{\partial}_h=\dfrac{\partial}{\partial \dot{x}^h}$.

The curvature tensor fields satisfy the identities

$$H^i_{jkh}+H^i_{khj}+H^i_{hjk}=0\ , \tag{1.3}$$

$$H^i_{jkh(l)}+H^i_{jhl(k)}+H^i_{jlk(h)}+H^r_{kh}G^i_{rjl}+H^r_{lk}G^i_{rjh}+H^r_{hl}G^i_{rjk}=0\ , \tag{1.4}$$

$$H^i_{jk(l)}+H^i_{kl(j)}+H^i_{lj(k)}=0 \tag{1.5}$$

and

$$H^i_{k(l)}-H^i_{l(k)}+H^i_{kl(r)}\dot{x}^r=0\ , \tag{1.6}$$

where index in the round bracket denotes covariant differentiation in the sense of *Berwald* [5].

Contracting H^i_{jk} and H^i_{jkh}, we obtain

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[1] Number in brackets refer to the references at the end of the paper.

(1.7) $$H^i_{ji}=H_j$$
and

(1.8) $$H^i_{jki}=H_{jk}=\dot{\partial}_j H_k$$

We have also

(1.9) $$\dot{x}^j H^i_{jkh}=H^i_{kh}\ .$$

(1.10) $$\dot{x}^j H^i_{jk}=H^i_k\ ,$$

(1.11) $$\dot{x}^j H_{jk}=H_k$$
and

(1.12) $$H=\frac{1}{n-1}H_{jk}\dot{x}^j\dot{x}^k\ .$$

The commutation formulae involving the curvature tensor fields are as follows:

(1.13) $$T_{(h)(k)}-T_{(k)(h)}=-\dot{\partial}_r T H^r_{hk}\ ,$$

(1.14) $$T^i_{j(h)(k)}-T^i_{j(k)(h)}=-\dot{\partial}_r T^i_j H^r_{hk}-T^i_r H^r_{jhk}+T^r_j H^i_{rhk}\ ,$$

(1.15) $$(\dot{\partial}_k T)_{(h)}-\dot{\partial}_k T_{(h)}=0$$
and

(1.16) $$(\dot{\partial}_k T^i_j)_{(h)}-\dot{\partial}_k T^i_{j(h)}=T^i_r G^r_{jkh}-T^r_j G^i_{rkh}\ .$$

A Finsler space F_n is said to be a recurrent Finsler space of first order, if the curvature tensor field H^i_{jkh} satisfies the relation

(1.17) $$H^i_{jkh(m)}=v_m H^i_{jkh}\ ,$$

where v_m is recurrence vector field.

Transvecting (1.17) successively by $\dot{x}$, we have

(1.18) $$H^i_{kh(m)}=v_m H^i_{kh}$$
and

(1.19) $$H^i_{h(m)}=v_m H^i_h\ .$$

Contracting (1.17), (1.18) and (1.19) with respect to the indices i and h, we get

(1.20) $$H_{jk(m)}=v_m H_{jk}\ ,$$

(1.21) $$H_{k(m)}=v_m H_k$$

and

(1.22) $$H_{(m)}=v_m H\ ,$$

respectively.

2. Recurrent curvature tensor fields of second order.

Definition 2.1. An n-dimensional Finsler space F_n in which the curvature tensor field H^i_{jkh} satisfies the relation

(2.1) $$H^i_{jkh(l)(m)}=a_{lm}H^i_{jkh}\ ,$$

where

(2.2) $$H^i_{jkh}\neq 0\ ,$$

is said to be a recurrent Finsler space of second order and a_{lm} is a recurrence tensor field. The curvature tensor fields defined in such a space are known as the recurrent curvature tensor fields of second order.

Transvecting (2.1) successively by $\dot{x}$, we obtain

(2.3) $$H^i_{kh(l)(m)}=a_{lm}H^i_{kh}$$

and

(2.4) $$H^i_{h(l)(m)}=a_{lm}H^i_h\ .$$

Contracting (2.1), (2.3) and (2.4) with respect to the indices i and h, we have

(2.5) $$H_{jk(l)(m)}=a_{lm}H_{jk}\ ,$$

(2.6) $$H_{k(l)(m)}=a_{lm}H_k$$

and

(2.7) $$H_{(l)(m)}=a_{lm}H\ ,$$

respectively.

Theorem 2.1. *The recurrence tensor field a_{lm} is non-symmetric.*

Proof. Commutating (2.5) in l and m, we have

(2.8) $$H_{jk(l)(m)}-H_{jk(m)(l)}=(a_{lm}-a_{ml})H_{jk}\ .$$

Multiplying (2.8) by $\dot{x}^j$ and $\dot{x}^k$ and noting (1.12), we get

(2.9) $$H_{(l)(m)}-H_{(m)(l)}=(a_{lm}-a_{ml})H\ .$$

By virtue of commutation formula (1.13), it becomes

(2.10) $$-\dot{\partial}_r H^r_{lm}=(a_{lm}-a_{ml})H\ ,$$

which proves the statement.

Theorem 2.2. *Every recurrent Finsler space for which the recurrence vector field v_m satisfies*

$$v_{m(l)}+v_m v_l \not\equiv 0 \, , \tag{2.11}$$

is a recurrent Finsler space of second order but the converse is not true in general.

Proof. The covariant differentiation of (1.17) gives

$$H^i_{jkh(m)(l)}=(v_{m(l)}+v_l v_m)H^i_{jkh} \, . \tag{2.12}$$

From (2.1) and (2.2), it yields

$$a_{ml}=(v_{m(l)}+v_m v_l) \, , \tag{2.13}$$

which proves the theorem.

From hereafter we shall consider such a recurrent Finsler space of second order and denote it by $\bar{F}_n$.

Theorem 2.3. *In $\bar{F}_n$, if the recurrence vector field is independent of $\dot{x}^i$, the recurrence tensor field a_{lm} is homogeneous of degree zero in $\dot{x}^i$.*

Proof. Differentiating (2.13) partially with respect to $\dot{x}^n$ and using the commutation formula (1.16), we have

$$\dot{\partial}_n a_{ml}=-v_r G^r_{mln} \, . \tag{2.14}$$

Multiplying (2.14) by $\dot{x}^n$, we get the result.

Theorem 2.4. *In $\bar{F}_n$, the recurrence tensor field a_{lm} satisfies the relation*

$$\begin{aligned}(a_{lm}-a_{ml})_{(n)}+(a_{mn}-a_{nm})_{(l)}+(a_{nl}-a_{ln})_{(m)} \\ =(a_{lm}-a_{ml})v_n+(a_{mn}-a_{nm})v_l+(a_{nl}-a_{ln})v_m \, .\end{aligned} \tag{2.15}$$

Proof. Commutating (2.3) with respect to the indices l and m and using the commutation formula (1.14), we have

$$(a_{lm}-a_{ml})H^i_{kh}=H^r_{hk}H^i_{rlm}-H^r_{lm}H^i_{rkh}-H^i_{rk}H^r_{hlm}-H^i_{rh}H^r_{klm} \, . \tag{2.16}$$

Differentiating (2.16) covariantly with respect to x^n and using (1.17), (1.18) and (2.16), we have

$$(a_{lm}-a_{ml})_{(n)}H^i_{kh}=v_n(a_{lm}-a_{ml})H^i_{kh} \, , \tag{2.17}$$

which yields

$$(a_{lm}-a_{ml})_{(n)}=v_n(a_{lm}-a_{ml}) \, . \tag{2.18}$$

Adding the expressions, obtained by cyclic change with respect to the indices l, m and n, we have the result.

Theorem 2.5. *In $\bar{F}_n$, the relation*

$$\begin{aligned}(2.19)\qquad &(\dot{\partial}_r a_{lm}-\dot{\partial}_r a_{ml})H^r_{ns}+(\dot{\partial}_r a_{ln}-\dot{\partial}_r a_{nl})H^r_{sm}+(\dot{\partial}_r a_{ls}-\dot{\partial}_r a_{sl})H^r_{mn}\\ &+(a_{mr}-a_{rm})H^r_{lns}+(a_{nr}-a_{rn})H^r_{lsm}+(a_{sr}-a_{rs})H^r_{lmn}\\ &=(a_{lm}-a_{ml})(v_{n(s)}-v_{s(n)})+(a_{ln}-a_{nl})(v_{s(m)}-v_{m(s)})\\ &+(a_{ls}-a_{sl})(v_{m(n)}-v_{n(m)})\ ,\end{aligned}$$

is true.

Proof. Differentiating (2.18) covariantly with respect to x^s and using (2.18), we get

$$(2.20)\qquad (a_{lm}-a_{ml})_{(n)(s)}=(a_{lm}-a_{ml})(v_{n(s)}+v_s v_n)\ .$$

Subtracting the result obtained by commutating (2.20) with respect to n and s, from (2.20), we get

$$(2.21)\qquad (a_{lm}-a_{ml})_{(n)(s)}-(a_{lm}-a_{ml})_{(s)(n)}=(a_{lm}-a_{ml})(v_{n(s)}-v_{s(n)})\ .$$

By virtue of the commutation formula (1.14), it becomes

$$\begin{aligned}(2.22)\qquad &(\dot{\partial}_r a_{lm}-\dot{\partial}_r a_{ml})H^r_{ns}+(a_{rl}-a_{lr})H^r_{mns}+(a_{mr}-a_{rm})H^r_{lns}\\ &=(a_{lm}-a_{ml})(v_{n(s)}-v_{s(n)})\ .\end{aligned}$$

Adding the expressions, obtained by cyclic change of (2.22) with respect to the indices m, n and s and using (1.3), we have Theorem 2.5.

Therorem 2.6. *In $\bar{F}_n$, the Bianchi identities satisfied by the curvature tensor fields, take the following forms*

$$\begin{aligned}(2.23)\qquad &(v_l v_s-a_{ls})H^i_{jkh}+(v_k v_s-a_{ks})H^i_{jhl}+(v_h v_s-a_{hs})H^i_{jlk}\\ &=H^r_{hk}G^i_{rjl(s)}+H^r_{lk}G^i_{rjh(s)}+H^r_{hl}G^i_{rjk(s)}\ ,\end{aligned}$$

$$(2.24)\qquad a_{hs}H^i_{jk}+a_{js}H^i_{kh}+a_{ks}H^i_{hj}=0$$

and

$$(2.25)\qquad a_{ls}H^i_k-a_{ks}H^i_l+a_{rs}H^i_{kl}\dot{x}^r=0\ .$$

Proof. Differentiating (1.4) covariantly with respect to x^s and using (1.18) and (2.1), we get

$$\begin{aligned}(2.26)\qquad &a_{ls}H^i_{jkh}+a_{ks}H^i_{jhl}+a_{hs}H^i_{jlk}=-v_s H^r_{kh}G^i_{rjl}-v_s H^r_{lk}G^i_{rjh}\\ &-v_s H^r_{hl}G^i_{rjk}-H^r_{kh}G^i_{rjl(s)}-H^r_{lk}G^i_{rjh(s)}-H^r_{hl}G^i_{rjk(s)}\ .\end{aligned}$$

By virtue of (1.4), (2.26) yields the result (2.23). The results (2.24) and (2.25) can be easily obtained by differentiating (1.5) and (1.6) covariantly and using (2.3) and (2.4).

Theorem 2.7. *In $\bar{F}_n$, the relation*

$$\dot{x}^m[H^i_{jk}\dot{\partial}_h v_{m(s)}+H^i_{hj}\dot{\partial}_k v_{m(s)}+H^i_{kh}\dot{\partial}_j v_{m(s)}]=0\,, \tag{2.27}$$

holds good.

Proof. Using the commutation formula (1.16) in the results obtained by differentiating (1.18) partially with respect to $\dot{x}^j$, we obtain

$$\dot{\partial}_j v_m H^i_{kh}=H^r_{kh}G^i_{rjm}-H^i_{rh}G^r_{kjm}-H^i_{kr}G^r_{hjm}\,. \tag{2.28}$$

Adding the expressions obtained by cyclic change of the indices j, k and h in (2.28), we obtain

$$(\dot{\partial}_j v_m)H^i_{kh}+(\dot{\partial}_k v_m)H^i_{hj}+(\dot{\partial}_h v_m)H^i_{jk}=H^r_{jk}G^i_{rhm}+H^r_{kh}G^i_{rjm}+H^r_{hj}G^i_{rkm}\,. \tag{2.29}$$

From (1.4), (2.29) gives

$$(\dot{\partial}_j v_m)H^i_{kh}+(\dot{\partial}_k v_m)H^i_{hj}+(\dot{\partial}_h v_m)H^i_{jk}=H^i_{mhk(j)}+H^i_{mjh(k)}+H^i_{mkj(h)}\,. \tag{2.30}$$

Differentiating (2.30) covariantly with respect to x^s and using (1.18) and (2.1), we get

$$\begin{aligned}(\dot{\partial}_j v_m)_{(s)}H^i_{kh}&+(\dot{\partial}_k v_m)_{(s)}H^i_{hj}+(\dot{\partial}_h v_m)_{(s)}H^i_{jk}+v_s(\dot{\partial}_j v_m)H^i_{kh}+v_s(\dot{\partial}_k v_m)H^i_{hj}\\ &+v_s(\dot{\partial}_h v_m)H^i_{jk}=a_{js}H^i_{mhk}+a_{ks}H^i_{mjh}+a_{hs}H^i_{mkj}\,.\end{aligned} \tag{2.31}$$

By virtue of the equations (1.16), (1.17) and (2.30), the equation (2.31) yields

$$\begin{aligned}(a_{js}-v_jv_s)H^i_{mhk}&+(a_{ks}-v_kv_s)H^i_{mjh}+(a_{hs}-v_hv_s)H^i_{mkj}\\ &=H^i_{kh}(\dot{\partial}_j v_{m(s)}+v_rG^r_{mjl})+H^i_{hj}(\dot{\partial}_k v_{m(s)}+v_rG^r_{mkl})+H^i_{jk}(\dot{\partial}_h v_{m(s)}+v_rG^r_{mhl})\,.\end{aligned} \tag{2.32}$$

Multiplying (2.32) by $\dot{x}^m$ and using (1.9) and (2.24), we have

$$v_s(v_hH^i_{jk}+v_kH^i_{hj}+v_jH^i_{kh})=\dot{x}^m[H^i_{jk}\dot{\partial}_h v_{m(s)}+H^i_{hj}\dot{\partial}_k v_{m(s)}+H^i_{kh}\dot{\partial}_j v_{m(s)}]\,. \tag{2.33}$$

By virtue of (1.5) and (1.18), (2.33) proves Theorem 2.7.

REFERENCES

[1] Mishra, R.S. and Pandey, H.D.: *Recurrent Finsler spaces*, J. Ind. Math. Soc., Vol. 32 (1968), 17-22.

[2] Sen, R.N.: *Finsler spaces of recurrent curvature*, Tensor N.S. Vol. 19 (1968), 291-299.

[3] Sinha, B.B. and Singh, S.P.: *On recurrent Finsler spaces* (to appear).

[4] Chaki, M.C. and Roy Chowdhary, A.N.: *On Ricci recurrent spaces of second order*, J. Ind. Math. Soc. Vol. 9, No. 2, (1967), 279-287.

[5] Rund, H.: *Differential geometry of Finsler spaces*, Springer-Verlag, (1959).

Department of Mathematics
Faculty of Science,
Banaras Hindu University
Varanasi-5, India