SOME INTEGRAL EQUATIONS INVOLVING CONFLUENT HYPERGEOMETRIC FUNCTIONS

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In the present paper, we use fractional integration to investigate a solution of the integral equation

$$(*) \qquad \int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} \Phi(a, c, t-x) f(t) dt = g(x), \ a>0, \ c>0 ,$$

where $\Phi(a, c, z)$ is the confluent hypergeometric functions. A necessary and sufficient condition is obtained for the existence of this solution.

A solution of the integral equation

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} \Phi(a, c, x-t) f(t) dt = g(x), \ a>0, \ c>a .$$

is found by reducing it to the equation (*).

1. Introduction. Recently several integral equations have been solved by means of fractional integration. *Erdélyi* [1] investigated the solutions of these integral equations whose kernels contain Legendre functions. *Love* [3] considered the integral equations involving hypergeometric functions and *Srivastava* [5] discussed the equations with polynomial kernels. *Wimp* [6] used the Laplace transform to solve an integral equation involving the confluent hypergeometric function. Here, we make use of the fractional integral operators to solve the integral equation

$$(1.1) \qquad \int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} \Phi(a, c, t-x) f(t) dt = g(x) , \quad (0<x<b<\infty) ,$$

where $\Phi(a, c, z)$ is the confluent hypergeometric function, $a>0$, $c>0$, g is a given function and f is to be determined.

2. Fractional integration. Let C_0 be the class of those continuous functions on the interval $(0, b)$, open at 0, where $0<b<\infty$, which are integrable at 0, and C_n, where n is a positive integer, be the class of all those functions which are n-times continuously differentiable on $(0, b)$ and which satisfy $f^{(k)}(0+)=0$, $k=0, 1, 2, \cdots, n-1$, while $f^{(n)}$ is integrable at 0.

Let I be the operator of integration defined by

(2.1) $$If(x)=\int_0^x f(t)dt\ , \quad (0<x<b<\infty)\ ,$$

so that the operation of k-times repeated integration is expressed as

(2.2) $$I^k f(x)=\int_0^x \frac{(x-t)^{k-1}}{(k-1)!} f(t)dt\ , \quad k=1, 2, \cdots ,$$

and set

(2.3) $$I^0 f(x)=f(x)\ , \quad I^{-k} f(x)=f^{(k)}(x)\ , \quad k=1, 2, \cdots .$$

The most important properties may be summerised as follows. For $f\in C_k$, $k=1, 2, 3, \cdots$, we have

(2.4) $$f(x)=I^k f^{(k)}(x)\ .$$

If $f\in C_j$, $j=1, 2, \cdots$, and k is an integer (positive negative or zero) for which $j+k\geq 0$, then $I^k f$ exists and belongs to C_{j+k}; if l is a further non-negative integer which does not exceed $j+k$, then

(2.5) $$\left(\frac{d}{dx}\right)^l I^k f(x)=I^{k-l} f(x)\ ,$$

exists and belongs to C_{j+k-l}.

We shall now extend these and other results to non-integral values of the index. For $\alpha>0$, we follow Riemann and Liouville in defining integration of order α as

(2.6) $$I^\alpha f(x)=\int_0^x \frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)} f(t)dt\ , \quad (0<x<b<\infty)\ .$$

Many authors have proved the existence almost everywhere of (2.6) for integrable functions. Under heavier restrictions upon f i.e. $f\in C_0$, $I^\alpha f$ exists and belongs to C_0.

For $\alpha>0$, $\beta>0$, $f\in C_0$,

(2.7) $$I^\alpha I^\beta f(x)=I^\beta I^\alpha f(x)=I^{\alpha+\beta} f(x)\ .$$

This can be proved by interchanging the order of integration in the repeated integral indicated on the left hand side of (2.7).

We now define $I^\alpha f$ for $\alpha<0$ as inverse operation to $I^{-\alpha}$, i.e. define $g=I^\alpha f$

for $\alpha<0$ and $f\in C_0$ to be the solution in C_0, if it exists, of the integral equation $f=I^{-\alpha}g$.

Hence, for any real α, the statement that $I^\alpha f$ exists implies that f and $I^\alpha f$ both belong to C_0. With this extension (2.7) holds for all real α and β (positive, negative or zero).

A sufficient condition for existence of $I^\alpha f$, where $\alpha<0$, is that $f\in C_k$ for some $k\geq-\alpha$; and

$$I^\alpha f(x)=I^{\alpha+k}f^{(k)}(x)\ . \tag{2.8}$$

Moreover, if j is a non-negative integer not exceeding $k+[\alpha]$, where $[\alpha]$ is the integral part of α, then

$$\left(\frac{d}{dx}\right)^j I^\alpha f(x)=I^{\alpha-j}f(x)=I^{\alpha-j+k}f^{(k)}(x)\ . \tag{2.9}$$

For $\alpha>0$, let us denote C_α the class of functions representable in the form $I^\alpha f$ with $f\in C_0$. This definition gives the class C_n of functions when $\alpha=n$. If $\alpha\geq0$, $\beta\geq0$ and $f\in C_\alpha$, then $I^\beta f\in C_{\alpha+\beta}$. If $0<\beta<\alpha$ and $f\in C_\alpha$, then $f\in C_\beta$, and it follows that for $0<\beta<\alpha$, $C_\alpha\subset C_\beta$. If $\alpha=-n$, n a positive integer, then $I^\alpha f$ exists and belongs to C_0 if and only if $f\in C_n$. If $\alpha=-n+\rho$, where n is a positive integer and $0<\rho<1$, then $f\in C_{n-1}$ is necessary and $f\in C_n$ is sufficient for the existence of $I^\alpha f\in C_0$, while a condition that is both necessary and sufficient is that $I^\rho f\in C_n$. Also if f and g are in C_0, then

$$I^\alpha f=I^\alpha g \quad \text{implies} \quad f=g\ . \tag{2.10}$$

This follows from *Kober*'s Uniqueness Theorem ([2]).

3. The integral equation. We first observe that if $\lambda>0$, $c>0$, $x>t$,

$$\int_t^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)}\,\frac{(s-t)^{c-1}}{\Gamma(c)}\,\Phi(a, c, t-s)ds=\frac{(x-t)^{c+\lambda-1}}{\Gamma(c+\lambda)}\,\Phi(a, c, t-x)\ . \tag{3.1}$$

Consider

$$\int_0^1 \frac{(1-u)^{\lambda-1}}{\Gamma(\lambda)}\,\frac{u^{c-1}}{\Gamma(c)}\,\Phi(a, c, zu)du=\frac{1}{\Gamma(c+\lambda)}\,\Phi(a, c+\lambda, z)\ .$$

Using

$$\Phi(a, c, z)=\sum_{r=0}^{\infty}\frac{(a)_r}{(c)_r}\,\frac{z^r}{r!}\ ,$$

where $(a)_r=a(a+1)\cdots(a+r-1)$ and changing the order of integration and summation and applying Beta Function we prove the last equation. Substitute in this equation

$$u=\frac{s-t}{x-t}, \quad z=t-x ,$$

to obtain (3.1). Set

$$H(a, c)f(x)=\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)}\Phi(a, c, t-x)f(t)dt , \tag{3.2}$$

then

$$I^{\lambda}H(a, c)f(x)=\int_0^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)}ds\int_0^s \frac{(s-t)^{c-1}}{\Gamma(c)}\Phi(a, c, t-s)f(t)dt .$$

Φ is bounded in the region of integration and so that order of integration may be changed by *Fubini's* theorem. Thus we get

$$I^{\lambda}H(a, c)f(x)=\int_0^x f(t)dt\int_t^x \frac{(x-s)^{\lambda-1}}{\Gamma(\lambda)}\,\frac{(s-t)^{c-1}}{\Gamma(c)}\Phi(a, c, t-s)ds ,$$

and by (3.1) we conclude that

$$I^{\lambda}H(a, c)f(x)=\int_0^x \frac{(x-t)^{c+\lambda-1}}{\Gamma(c+\lambda)}\Phi(a, c+\lambda, t-x)f(t)dt ,$$

$$I^{\lambda}H(a, c)f(x)=H(a, c+\lambda)f(x) . \tag{3.3}$$

Now we turn to the integral equation (1.1). Using (3.2) we rewrite (1.1) as

$$H(a, c)f(x)=g(x), \quad a>0, \quad c>0.$$

Then

$$I^{a}H(a, c)f(x)=I^{a}g(x) ,$$

$$H(a, c+a)f(x)=I^{a}g(x) ,$$

$$I^{c}H(a, a)f(x)=I^{a}g(x) ,$$

$$H(a, a)f(x)=I^{-c}I^{a}g(x) .$$

This may be written as

$$\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}\Phi(a, a, t-x)f(t)dt=I^{-c}I^{a}g(x) .$$

Since $\Phi(a, a, z)=e^{z}$, we obtain

$$\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}\cdot e^{t-x}f(t)dt=I^{-c}I^{a}g(x) ,$$

$$I^a(e^x f(x)) = e^x I^{-c} I^a g(x) .$$

Hence

(3.4) $$f(x) = e^{-x} I^{-a} e^x I^{-c} I^a g(x) .$$

This is a solution of the integral equation if it exists. Also (3.4) implies that

(3.5) $$g(x) = I^{-a} I^c e^{-x} I^a e^x f(x) .$$

4. Necessary and sufficient conditions. Before we discussed the necessary and sufficient condition for the existence of the solution, in C_0, of the integral equation (1.1), we prove the following result.

If $a>0$, $0<x<b<\infty$, *then*

(4.1) $$I^a e^x f(x) = e^x I^a g(x) ,$$

has, for each function $f \in C_0$, *a solution* g *which also belongs to* C_0; *and for each* g *in* C_0, *a solution* f *which also belongs to* C_0.

To prove it we first show that for $c>0$, $0<t<x<\infty$,

(4.2) $$\frac{(x-t)^{c-1}}{\Gamma(c)} [\Phi(a, c, t-x) - 1] = -a \int_t^x \frac{(x-s)^{c-1}}{\Gamma(c)} \Phi(a+1, 2, t-s) ds .$$

In the equation

(4.3) $$az \int_0^1 \frac{(1-u)^{c-1}}{\Gamma(c)} \Phi(a+1, 2, zu) du = \frac{1}{\Gamma(c)} [\Phi(a, c, z) - 1] .$$

Substitute

$$z = t-x, \quad u = \frac{s-t}{x-t} ,$$

to get (4.2). Now assume $f \in C_0$, then

$$\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} (1-e^{t-x}) f(t) dt = \int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} [1-\Phi(a, a, t-x)] f(t) dt$$
$$= a \int_0^x f(t) dt \int_t^x \frac{(x-s)^{a-1}}{\Gamma(a)} \Phi(a+1, 2, t-s) ds, \quad \text{by (2)} .$$

Changing the order of integration which is permissible by *Fubini*'s theorem since Φ bounded in the region of integration, we obtain

$$\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} f(t) dt - \int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)} e^{t-x} f(t) dt$$
$$= a \int_0^x \frac{(x-s)^{a-1}}{\Gamma(a)} ds \int_0^s \Phi(a+1, 2, t-s) f(t) dt .$$

Thus

$$e^{-x}I^a e^x f(x)=I^a[f(x)+\int_0^x \Phi(a+1, 2, t-x)]f(t)dt\ .$$

This shows that (4.1) is satisfied by

$$g(x)=f(x)-a\int_0^x \Phi(a+1, 2, t-x)f(t)dt\ .$$

Before we consider the second part of the above result we substitute in (4.3)

$$z=x-t\ ,\quad u=\frac{s-t}{x-t}\ .$$

to get

$$\frac{(x-t)^{c-1}}{\Gamma(c)}[\Phi(a, c, x-t)-1]=a\int_t^x \frac{(x-s)^{c-1}}{\Gamma(a)}\Phi(a+1, 2, s-t)ds\ , \tag{4.4}$$

for $c>0$, $0<x<b<\infty$.

Now suppose $f\in C_0$, then

$$\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}(e^{x-t}-1)f(t)dt=\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}[\Phi(a, a, x-t)-1]f(t)dt$$
$$=a\int_0^x f(t)dt\int_t^x \frac{(x-s)^{a-1}}{\Gamma(a)}\Phi(a+1, 2, s-t)ds\ ,$$

by (3.4). Inversion of order of integration which is permissible by *Fubini's* theorem since Φ is bounded in the region of integration, yields

$$e^x\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}e^{-t}f(t)dt-\int_0^x \frac{(x-t)^{a-1}}{\Gamma(a)}f(t)dt$$
$$=a\int_0^x \frac{(x-s)^{a-1}}{\Gamma(a)}ds\int_0^s \Phi(a+1, 2, s-t)f(t)dt\ .$$

Thus

$$e^x I^a e^{-x}f(x)=I^a g(x)\ ,$$

where

$$g(x)=f(x)+a\int_0^x \Phi(a+1, 2, t-x)f(t)dt\ ,$$

and g belongs to C_0. Hence for $f\in C_0$, there is $g\in C_0$ such that

$$I^a e^{-x}f(x)=e^{-x}I^a g(x)\ . \tag{4.5}$$

To prove second part of the theorem assume $G\in C_0$. Let $f(x)=e^x G(x)$. Then

$f \in C_0$ and there exists a function $g \in C_0$, by (4.5), such that

$$I^a e^{-x} f(x) = e^{-x} I^a g(x) .$$

Let $e^{-x}g(x)=F(x)$. Then $F \in C_0$ and we have

$$I^a G(x) = e^{-x} I^a e^x F(x) . \tag{4.6}$$

Thus, given $G \in C_0$, there exists a function $F \in C_0$ such that (4.6) is true.

We, now, prove that necessary and sufficient condition for the existence of solution, in C_0, of the integral equation (1.1) is that $g \in C_c$.

Indeed, let $f \in C_0$, then using (4.1) we get

$$I^a g(x) = I^c e^{-x} I^a e^x f(x) ,$$

$$= I^c I^a h(x) ,$$

where $h \in C_0$.

$$I^a g(x) = I^a I^c h(x) ,$$

by (2.7). Thus an application (2.10) yields

$$g(x) = I^c h(x) .$$

Hence $g \in C_c$ is a necessary condition.

Now assume that $g \in C_c$, then

$$g(x) = I^c h(x) ,$$

where $h \in C_0$.

$$I^a g(x) = I^a I^c h(x) = I^c I^a h(x) ,$$

and using (3.5) we rewrite as

$$I^c e^{-x} I^a e^x f(x) = I^c I^a h(x) = I^c e^{-x} I^a e^x u(x) ,$$

where $u \in C_0$. By successive applications of (2.10) we finally have

$$f(x) = u(x) .$$

Hence $g \in C_c$ is a sufficient condition.

5. Another integral equation. We now deduce a solution of the integral equation

$$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} \Phi(a, c, x-t) f(t) dt = g(x) , \tag{5.1}$$

for $c>0$, $c>a$, $0<x<b<\infty$. Using the *Kummer's* relation ([4], p. 125)

(5.2) $$\Phi(a, c, z)=e^{z}\Phi(c-a, c, -z) ,$$

we obtain

(5.3) $$\int_0^x \frac{(x-t)^{c-1}}{\Gamma(c)} \Phi(c-a, c, t-x)e^{-t} f(t)dt=e^{-x}g(x), \quad (0<x<b<\infty) ,$$

for $c>0$, $c>a$. Thus if $c>0$, $c>a$, $f\in C_0$,

(5.4) $$f(x)=I^{a-c}e^{x}I^{-c}I^{c-a}e^{-x}g(x) .$$

Also we have

(5.5) $$g(x)=e^{x}I^{a-c}I^{c}e^{-x}I^{c-a}f(x) .$$

To determine necessary and sufficient condition for the existence of solution (5.4), in C_0, of the integral equation (5.1) we prove the following result.

If $a>0$, $0<x<b<\infty$, *then*

(5.6) $$I^{a}e^{-x}f(x)=e^{-x}I^{a}g(x) ,$$

has, for each $f\in C_0$, *a solution* g *in* C_0 *which also belongs to* C_0; *and for each* g *in* C_0, *a solution* f *which also belongs to* C_0.

First part is proved in (4.5). To prove the second part of (5.6) assume $G\in C_0$. Let $f(x)=e^{-x}G(x)$. By (4.1), there is $g\in C_0$ such that

$$I^{a}G(x)=e^{x}I^{a}g(x) .$$

Let $e^{x}g(x)=F(x)$. If $g\in C_0$, then $F\in C_0$ and we have

$$I^{a}G(x)=e^{x}I^{a}e^{-x}F(x) .$$

Hence given $G\in C_0$ there is a function $F\in C_0$ such that (5.6) holds.

We shall, now, prove that necessary and sufficient condition for the existence of solution, in C_0, of the integral equation (5.1) is that $g\in C_c$.

Indeed, let $f\in C_c$ then using (5.6) we get

$$I^{c-a}e^{x}g(x)=I^{c}e^{-x}I^{c-a}f(x) ,$$

$$=I^{c}I^{c-a}e^{-x}u(x) ,$$

where $u\in C_0$. Also use of (2.7) and (2.10) gives

$$e^{-x}g(x)=I^{c}e^{-x}u(x) .$$

Another application of (5.6) yields

$$g(x)=I^{c}v(x) ,$$

where $v\in C_0$. Hence $g\in C_c$ is necessary condition.

Now suppose that $g(x)=I^c\psi(x)$, $\psi\in C_0$. Then

$$I^c e^{-x} I^{c-a} f(x) = I^{c-a} e^{-x} I^c \psi(x) ,$$

$$= I^{c-a} I^c e^{-x} u(x) , \quad \text{by (5.6)} ,$$

where $u\in C_0$. Using (2.7) and (2.10) we obtain

$$e^{-x} I^{c-a} f(x) = I^{c-a} e^{-x} u(x) ,$$

$$= e^{-x} I^{c-a} v(x) , \quad \text{by (5.6)} ,$$

where $v\in C_0$. Hence another application of (2.10) yields

$$f(x)=v(x) .$$

Hence $g\in C_c$ is sufficient condition.

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