ON NON-CONTRACTIVE MAPPING

By

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Let (X, d) be a metric space and f a mapping of X into itself. The well-known Banach's fixed point theorem states as follows; If (X, d) is complete and if $d(f(x), f(y)) \leqq kd(x, y)$ for all $x, y \in X$ where $k \in (0, 1)$, then there exists a unique fixed point $z \in X$ and for each $x \in X$, the sequence of iterates $\{f^n(x)\}$ converges to z.

Kannan proved that if (X, d) is complete and if $d(f(x), f(y)) \leqq k\{d(x, f(x)) + d(y, f(y))\}$ for all $x, y \in X$ where $k \in \left(0, \frac{1}{2}\right)$, then the same result is obtained.

The above two theorems are independent.

In this paper, we will give a more general theorem and an equivalent condition to the conclusion of Banach's theorem.

1. Theorem 1. *If (X, d) is complete and if there exist real-valued functions α, β and γ defined on a direct product set $X \times X$ which are symmetric and bounded, and for every x, y in X, there exist positive integers $M(x)$, $M(y)$ such that for $p=0$ and $p=1$,*

$$\alpha(x, y)d(f^{M(x)}(x), f^{M(y)+p}(y)) + \beta(x, y)d(x, y) \leqq \gamma(x, y)\{d(x, f^{M(x)}(x)) + d(y, f^{M(y)+p}(y))\}$$

where $\quad \beta(x, y) < \gamma(x, y) \leqq \dfrac{k\alpha(x, y) + \beta(x, y)}{1+k}$, $k \in (0, 1)$ *and* $0 < s \leqq \alpha(x, y) - \gamma(x, y)$,

then there exists at least one fixed point.

Proof. Let x be arbitrary. Put $x_0 = x$ and $x_{n+1} = f^{M(x_n)}(x_n)$ where $n = 0, 1, 2, \cdots$. By the condition,

$$d(x_n, x_{n+1}) \leqq \left\{\frac{\gamma(x_{n-1}, x_n) - \beta(x_{n-1}, x_n)}{\alpha(x_{n-1}, x_n) - \gamma(x_{n-1}, x_n)}\right\} d(x_{n-1}, x_n) \leqq kd(x_{n-1}, x_n) ,$$

and so $d(x_n, x_{n+1}) \leqq k^n d(x_0, x_1)$. Then it is easily seen that $\{x_n\}$ is a Cauchy sequence. Therefore $\lim_{n\to\infty} x_n = z$ for some $z \in X$ since (X, d) is complete. Again by the given inequality, we have

$$\{\alpha(x_n, z) - \gamma(x_n, z)\} d(z, f^{M(z)}(z) \leqq \alpha(x_n, z)\{d(z, f^{M(z)}(z)) - d(x_{n+1}, f^{M(z)}(z))\}$$
$$- \beta(x_n, z)d(x_n, z) + \gamma(x_n, z)d(x_n, x_{n+1}) .$$

Thus $f^{M(z)}(z)=z$ since $0<s\leqq\alpha(x_n, z)-\gamma(x_n, z)$ and since α, β and γ are bounded. Taking $x=y=z$ and $p=1$ in the given inequality, we obtain that $f(z)=z$.

Q.E.D.

The following example shows that Theorem 1 does not conclude that for each $x\in X$, the sequence of iterates $\{f^n(x)\}$ converges to a fixed point, and shows too that the fixed point is not necessarily unique in Theorem 1.

Example 1. Let $X=\{-1\}\cup\{0\}\cup X_1\cup X_2$, where $X_1=\left\{\left(\frac{1}{2}\right)^n; n=1, 2, 3, \cdots\right\}$ and $X_2=\{2n+1; n=0, 1, 2, \cdots\}$ with the usual distance. Define a mapping f by;

(1) $$f(-1)=-1 \text{ and } f(0)=0,$$

(2) $$f\left(\left(\frac{1}{2}\right)^{2n+1}\right)=2n+1; n=0, 1, 2, \cdots,$$

(3) $$f((2n+1))=\left(\frac{1}{2}\right)^{2(n+1)}; n=0, 1, 2, \cdots,$$

(4) $$f\left(\left(\frac{1}{2}\right)^{2n}\right)=\left(\frac{1}{2}\right)^{2n+1}; n=1, 2, 3, \cdots.$$

Then we may take, in this example,

(i) $M(2n+1)=1$ where $n=0, 1, 2, \cdots$, $M\left(\left(\frac{1}{2}\right)^{2n}\right)=3$ where $n=1, 2, 3, \cdots$, and $M\left(\left(\frac{1}{2}\right)^{2n+1}\right)=5$ where $n=0, 1, 2, \cdots$,

(ii) $\alpha(x, y)=1$, $\beta(x, y)=0$ and $\gamma(x, y)=\frac{1}{3}$ if $x, y\in X-\{-1\}$,

(iii) $\alpha(-1, x)=1$, $\beta(-1, x)=-\frac{5}{8}$ and $\gamma(-1, x)=0$ if $x\in X_2$,

(iv) $\alpha(-1, x)=1$, $\beta(-1, x)=-1$ and $\gamma(-1, x)=-\frac{1}{2}$ if $x\in X_1$ or if $x=0$.

Corollary 1. *If $M(x)=M(y)=1$ for every $x, y\in X$ and $p=0$ in the condition of Theorem 1, then there exists at least one fixed point and the sequence of iterates $\{f^n(x)\}$ converges to a fixed point for each $x\in X$.*

The following corollary gives a necessary and sufficient condition for the conclusion of Banach fixed point theorem.

Corollary 2. *When (X, d) is complete, the following two statements are equivalent.*

(i) *There exists a unique fixed point and the sequence of iterates $\{f^n(x)\}$ converges to the fixed point for each $x\in X$.*

(ii) *For some $k\in\left(0, \frac{1}{2}\right)$ and each x, y in X, there exist positive integers $M(x)$, $M(y)$ such that*

$$d(f^{M(x)}(x), f^p(y))\leqq k\{d(x, f^{M(x)}(x))+d(y, f^p(y))\} \text{ whenever } p\geqq M(y).$$

Proof. (i)$\Longrightarrow$(ii). Let z be the unique fixed point, and $k\in\left(0, \frac{1}{2}\right)$ given and let x, y be arbitrary. Since $\lim_{n\to\infty} f^n(x) = \lim_{n\to\infty} f^n(y) = z$, there exist positive integers $M(x)$ and $M(y)$ such that

$$d(z, f^m(x)) \leqq \frac{k}{1+k} d(z, x) \quad \text{and} \quad d(z, f^n(y)) \leqq \frac{k}{1+k} d(z, y)$$

whenever $m \geqq M(x)$ and $n \geqq M(y)$ respectively. Thus $d(z, f^m(x)) \leqq kd(x, f^m(x))$ and $d(z, f^n(y)) \leqq kd(y, f^n(y))$ whenever $m \geqq M(x)$ and $n \geqq M(y)$ respectively. Since $d(f^m(x), f^n(y)) \leqq d(z, f^m(x)) + d(z, f^n(y))$, (ii) is obtained. Conversely (ii)$\Longrightarrow$(i). Let x be arbitrary. Put $x_0 = x$ and $x_{n+1} = f^{M(x_n)}(x_n)$ where $n = 0, 1, 2, \cdots$. By Theorem 1, there exists a fixed point z and $\lim_{n\to\infty} x_n = z$. It is easily seen that the fixed point is unique. Next, for all $p \geqq M(x_n)$,

$$\begin{aligned} d(z, f^p(x_n)) &= d(f^{M(z)}(z), f^p(x_n)) \leqq k\{d(z, f^{M(z)}(z)) + d(x_n, f^p(x_n))\} \\ &= kd(x_n, f^p(x_n)) \leqq k\{d(z, x_n) + d(z, f^p(x_n))\} . \end{aligned}$$

Thus $d(z, f^p(x_n)) \leqq \dfrac{k}{1-k} d(z, x_n)$ for all $p \geqq M(x_n)$. Hence $\lim_{n\to\infty} f^n(x) = z$. Q.E.D.

Corollary 3. *If the condition of Theorem 1 is satisfied in the case $p=0$ only, then there exists a periodic point z, i.e. $f^{M(z)}(z) = z$.*

The following example shows that in Corollary 3, the periodic point is not necessarily a fixed point.

Example 2. Let $X = \left[\frac{1}{2}, \frac{2}{3}\right] \cup \left[\frac{3}{2}, 2\right]$ with the usual distance. Define a mapping f by $f(x) = \dfrac{1}{x}$ for all $x \in X$. Then we may take $M(x) = 2$ for each $x \in X$, and $\alpha(x, y) = 1$, $\beta(x, y) = -4$ and $\gamma(x, y) = -3$ for every $x, y \in X$.

2. Theorem 2. *If* (i) *there exist real-valued functions α, β and γ defined on a direct product set $X \times X$ which are symmetric and bounded,*

(ii) $0 < s \leqq \alpha(x, y) - \gamma(x, y)$ *for all* $x, y \in X$,

(iii) *for each $x \in X$, there exist a positive integer $M(x)$ and a positive number $\delta(x)$ such that for every $y \in \{y;\, d(x, y) < \delta(x)\}$,*

$$\alpha(x, y) d(f^{M(x)}(x), f^p(y)) + \beta(x, y) d(x, y) \leqq \gamma(x, y)\{d(x, f^{M(x)}(x)) + d(y, f^p(y))\}$$

whenever $p \geqq M(y)$ where $M(y)$ is the positive integer corresponding to y,

(iv) *for some $x \in X$, the sequence of iterates $\{f^n(x)\}$ has a subsequence $\{f^{n_i}(x)\}$ which converges to a point $z \in X$, then $f(z) = z$ and $\lim_{n\to\infty} f^n(x) = z$.*

Proof. By (iv) there exists a positive integer $N(z, \delta(z))$ such that $d(z, f^{n_i}(x)) < \delta(z)$ whenever $i \geqq N(z, \delta(z))$.

Let

$$p_i = \min\{p;\, M(f^{n_i}(x)) \leqq p,\ f^{p+n_i}(x) = f^{n_j}(x) \in \{f^{n_i}(x)\}\} .$$

Then by (iii),

$$\alpha(z, f^{n_i}(x))d(f^{M(z)}(z),\ f^{p_i+n_i}(x))+\beta(z, f^{n_i}(x))d(z, f^{n_i}(x))$$
$$\leqq \gamma(z, f^{n_i}(x))\{d(z, f^{M(z)}(z))+d(f^{n_i}(x), f^{p_i+n_i}(x))\}\ .$$

Thus

$$\alpha(z, f^{n_i}(x))d(f^{M(z)}(z), f^{n_j}(x))+\beta(z, f^{n_i}(x))d(z, f^{n_i}(x))$$
$$\leqq \gamma(z, f^{n_i}(x))\{d(z, f^{M(z)}(z))+d(f^{n_i}(x), f^{n_j}(x))\}\ .$$

and so

$$\{\alpha(z, f^{n_i}(x))-\gamma(z, f^{n_i}(x))\}d(z, f^{M(z)}(z)\leqq \alpha(z, f^{n_i}(x))\{d(z, f^{M(z)}(z)-d(f^{M(z)}(z), f^{n_j}(x))\}$$
$$-\beta(z, f^{n_i}(x))d(z, f^{n_i}(x))+\gamma(z, f^{n_i}(x))d(f^{n_i}(x), f^{n_j}(x))\ .$$

Hence $f^{M(z)}(z)=z$. Taking $x=y=z$ and $p=M(z)+1$ in (iii), we obtain $f(z)=z$. Finally, taking $x=z$ and $y=f^{n_i}(x)$ in (iii), we have

$$\alpha(z, f^{n_i}(x))d(z, f^{p+n_i}(x))+\beta(z, f^{n_i}(x))d(z, f^{n_i}(x))\leqq \gamma(z, f^{n_i}(x))d(f^{n_i}(x), f^{p+n_i}(x))$$

for all $p\geqq M(f^{n_i}(x))$.

Let $K=\max\{|\alpha|, |\beta|, |\gamma|\}$. There exist following three cases.

(1) When $0<\gamma(z, f^{n_i}(x))$, since

$d(f^{n_i}(x), f^{p+n_i}(x))\leqq d(z, f^{n_i}(x))+d(z, f^{p+n_i}(x))$, we have

$$d(z, f^{p+n_i}(x))\leqq \left\{\frac{\gamma(z, f^{n_i}(x))-\beta(z, f^{n_i}(x))}{\alpha(z, f^{n_i}(x))-\gamma(z, f^{n_i}(x))}\right\}d(z, f^{n_i}(x))\leqq \frac{2K}{s}\, d(z, f^{n_i}(x))$$

for all $p\geqq M(f^{n_i}(x))$.

(2) When $\gamma(z, f^{n_i}(x))=0$, since $0<s\leqq\alpha(z, f^{n_i}(x))$, we have

$$d(z, f^{p+n_i}(x))\leqq -\left\{\frac{\beta(z, f^{n_i}(x))}{\alpha(z, f^{n_i}(x))}\right\}d(z, f^{n_i}(x))\leqq \frac{K}{s}\, d(z, f^{n_i}(x))$$

for all $p\geqq M((f^{n_i}(x))$.

(3) When $\gamma(z, f^{n_i}(x))<0$, since

$d(z, f^{p+n_i}(x))\leqq d(z, f^{n_i}(x))+d(f^{n_i}(x), f^{p+n_i}(x))$, we have

$$d(z, f^{p+n_i}(x))\leqq -\left\{\frac{\beta(z, f^{n_i}(x))+\gamma(z, f^{n_i}(x))}{\alpha(z, f^{n_i}(x))-\gamma(z, f^{n_i}(x))}\right\}d(z, f^{n_i}(x))\leqq \frac{2K}{s}\, d(z, f^{n_i}(x))$$

for all $p\geqq M(f^{n_i}(x))$.

Therefore $\lim_{n\to\infty} f^n(x)=z$. Q.E.D.

The following corollary is corresponding to Corollary 2 in the case of compact metric spaces.

Corollary 4. *Let (X, d) be compact. Then the following two statements are equivalent.*

(i) *There exists a unique fixed point and the sequence of iterates* $\{f^n(x)\}$ *converges to the fixed point for each* $x \in X$.

(ii) *For some* $k \in (0,1)$ *and for every* x, y *in* X, *there exist positive integers* $M(x)$, $M(y)$ *such that*

$$d(f^{M(x)}(x), f^p(y)) \leqq k\{d(x, f^{M(x)}(x)) + d(y, f^p(y))\} \text{ whenever } p \geqq M(y) .$$

Proof. (i)$\Longrightarrow$(ii). It can be proved in the same way as is seen in the proof of Corollary 2. Conversely (ii)$\Longrightarrow$(i). Since (X, d) is compact, the sequence of iterates $\{f^n(x)\}$ has a subsequence which converges to a point $z \in X$ for each $x \in X$. By Theorem 2, $f(z)=z$ and $\lim_{n\to\infty} f^n(x)=z$. It is easily seen that the fixed point is unique. Q.E.D.

In [3], we proved that if (i) for each $x \in X$, $d(O(x))<\infty$ and $\lim_{n\to\infty} d(O(f^n(x)))$ $<d(O(x))$ when $0<d(O(x))$, where $O(f^p(x))=\bigcup_{n=p}^{\infty}\{f^n(x)\}$, (ii) for each $x \in X$ and every positive number ε, there exist a positive integer $M(x)$ and a positive number $\delta(x, \varepsilon)$ such that $d(x, y)<\delta(x, \varepsilon)$ implies $d(f^p(x), f^p(y))<\varepsilon$ for every $p \geqq M(x)$, (iii) for some $x \in X$, the sequence of iterates $\{f^n(x)\}$ has a subsequence which converges to a point $z \in X$, then $f(z)=z$ and $\lim_{n\to\infty} f^n(x)=z$.

The following example shows that Theorem 2 does not imply the above statement.

Example 3. Let $X=\{0\}\cup X_1\cup X_2\cup X_3$, where $X_1=\left\{\left(\frac{1}{2}\right)^n; n=1, 2, 3, \cdots\right\}$, $X_2=\left\{\left(\frac{1}{3}\right)^n; n=1, 2, 3, \cdots\right\}$ and $X_3=\{n; n=1, 2, 3, \cdots\}$, with the usual distance. Define a mapping f by;

(1) $$f(0)=0 \text{ and } f(1)=\frac{1}{2}$$

(2) $$f\left(\left(\frac{1}{2}\right)^n\right)=\left(\frac{1}{2}\right)^{n+1} \text{ and } f\left(\left(\frac{1}{3}\right)^n\right)=n;\ n=1, 2, 3, \cdots$$

(3) $$f(n)=n-1 \text{ if } n \geqq 2.$$

In this example, for each $x \in X$, $\lim_{n\to\infty} f^n(x)=0$ and 0 is a unique fixed point. Therefore by the proof of Corollary 2 or 4, the condition of theorem 2 is satisfied. But there does not exist a positive integer $M(0)$ in the above statement.

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