THE DECOMPOSITION THEOREMS FOR VECTOR MEASURES

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Abstract. In this paper we shall prove the following (1) and (2).
(1) Let S be a set, γ a σ-ring of subsets of S, X a normed space, $m; \gamma \to X$ a vector measure and μ a non-negative measure on γ. Then there exist unique m_0 and m_1 such that $m=m_0+m_1$, where $m_0 \ll \mu$ and $m_1 \perp \mu$ (Theorem 1).
(2) Let S be a locally compact Hausdorff space, $\mathfrak{B}(S)$ the σ-ring generated by the compact subsets of S, X a normed space and $m; \mathfrak{B}(S) \to X$ a Borel measure. Then there exist unique regular m_0 and antiregular m_1 such that $m=m_0+m_1$ (Theorem 3).

1. Introduction

In [1], [2] *Johnson* has proved the following (1) and (2).
(1) Let S be a set, γ a σ-ring of subsets of S, ν a strongly σ-finite, non-negative measure on γ and μ a non-negative measure on γ. Then there exist unique ν_0 and ν_1 such that $\nu=\nu_0+\nu_1$, where $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$ ([1] Theorem 3.4).
(2) Let S be a locally compact Hausdorff space, $\mathfrak{B}(S)$ the σ-ring generated by the compact subsets of S and μ a Borel measure on $\mathfrak{B}(S)$. Then there exist unique regular μ_0 and antiregular μ_1 such that $\mu=\mu_0+\mu_1$ ([2] Theorem 2.3).

In this paper we shall extend these results to the case of vector measures.

2. The Lebesgue decomposition theorem

Let S be a set, γ a σ-ring of subsets of S and X a normed space.

Definition 1. A set function m defined on γ with values in X is called a vector measure if for every sequence $\{E_n\}$ of mutually disjoint sets of γ we have $m(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} m(E_n)$.

Definition 2. Let $m; \gamma \to X$ be a vector measure and μ a non-negative measure on γ. m is called absolutely continuous with respect to μ ($m \ll \mu$) if for every set $A \in \gamma$ such that $\mu(A)=0$ we have $m(A)=0$.

Proposition 1. *$m \ll \mu$ if and only if for every number $\varepsilon>0$ there exists a number $\delta=\delta(\varepsilon)>0$ such that for every $A \in \gamma$ with $\mu(A)<\delta$ we have $\|m(A)\|<\varepsilon$.*

Proof. The sufficiency is clear.

Necessity. If it is false, then there exists a number $\varepsilon_0>0$ such that for every number $\delta>0$ there exists a set $A_\delta\in\gamma$ such that $\mu(A_\delta)<\delta$ and $\|m(A_\delta)\|\geqq\varepsilon_0$. Taking $\delta=1/2^n$ and $A_n=A_\delta$ we have $\mu(A_n)<1/2^n$ and $\|m(A_n)\|\geqq\varepsilon_0$ for all n. If we put $B_n=\bigcup_{k=n}^{\infty}A_k$ and $B=\bigcap_{n=1}^{\infty}B_n$, then $\mu(B)\leqq\sum_{k=n}^{\infty}\mu(A_k)\leqq 1/2^{n-1}$ for all n. Hence $\mu(B)=0$. For every $C\in\gamma$ with $C\subset B$ $\mu(C)=0$. Since $m\ll\mu$, $m(C)=0$. We put $\tilde{m}(E)=\sup\{\|m(A)\|;\ A\subset E,\ A\in\gamma\}$. Then $\tilde{m}(B)=0$. On the other hand $\tilde{m}(B_n)\geqq\tilde{m}(A_n)\geqq\|m(A_n)\|\geqq\varepsilon_0$ for all n. Since $\{B_n\}$ is a decreasing sequence, by Gould ([5] Corollary 3.6. In this case, σ-field can be replaced by a σ-ring) we have $\tilde{m}(B)=\lim_{n\to\infty}\tilde{m}(B_n)\geqq\varepsilon_0$. Therefore we get a contradiction.

Proposition 2. *For any vector measure $m;\gamma\to X$ there exists a finite non-negative measure ν on γ such that*

(1) $m\ll\nu$.

(2) $\nu(E)\leqq\tilde{m}(E)=\sup\{\|m(A)\|;\ A\subset E,\ A\in\gamma\}$ *for every* $E\in\gamma$.

Proof. See *Dinculeanu* and *Kluvanek* ([4] Theorem 1).

Definition 3. (1) A set $A\subset S$ is called locally measurable if $A\cap E\in\gamma$ for every $E\in\gamma$.

(2) Let $m;\gamma\to X$ be a vector measure and μ a non-negative measure on γ. m is singular with respect to μ ($m\perp\mu$) if there exists a locally measurable set A such that $m(E\cap A)=0$ and $\mu(E-A)=0$ for every $E\in\gamma$.

Lemma 1. *Any vector measure $m;\gamma\to X$ which is both $m\ll\mu$ and $m\perp\mu$ is the zero measure.*

Proof. Since $m\perp\mu$, there exists a locally measurable set A such that $m(E\cap A)=0$ and $\mu(E-A)=0$ for every $E\in\gamma$. Since $m\ll\mu$ we have $m(E-A)=0$. Hence $m(E)=m(E\cap A)+m(E-A)=0$ for every $E\in\gamma$. Therefore m is the zero measure.

Theorem 1. *Let $m;\gamma\to X$ be a vector measure and μ a non-negative measure on γ. Then there exist unique m_0 and m_1 such that $m=m_0+m_1$, where $m_0\ll\mu$ and $m_1\perp\mu$.*

Proof. By Proposition 2 there exists a finite non-negative measure ν on γ such that $m\ll\nu$. Since ν is finite, ν is strongly σ finite ([1] p. 631).

By *Johnson* ([1] Theorem 3.4) there exist unique ν_0 and ν_1 such that $\nu=\nu_0+\nu_1$, where $\nu_0\ll\mu$ and $\nu_1\perp\mu$. Since $\nu_1\perp\mu$ there exists a locally measurable set $A\subset S$ such that $\nu_1(E\cap A)=0=\mu(E-A)$ for every $E\in\gamma$. Since $\nu_0\ll\mu$ we have $\nu_0(E-A)=0$. Hence $\nu(E\cap A)=\nu_0(E\cap A)+\nu_1(E\cap A)=\nu_0(E\cap A)=\nu_0(E\cap A)+\nu_0(E-A)=\nu_0(E)$ and $\nu(E-A)=\nu_0(E-A)+\nu_1(E-A)=\nu_1(E-A)=\nu_1(E\cap A)+\nu_1(E-A)=\nu_1(E)$. We put $m_0(E)=m(E\cap A)$ and $m_1(E)=m(E-A)$ for every $E\in\gamma$. Then m_0 and m_1 are vector measure and $m(E)=m(E\cap A)+m(E-A)=m_0(E)+m_1(E)$. If $\mu(E)=0$, then $\nu_0(E)=\nu(E\cap A)=0$. Since $m\ll\nu$, we have $m_0(E)=m(E\cap A)=0$, which shows $m_0\ll\mu$. Since $\nu_1\perp\mu$, there exists a locally measurable set $A\subset S$ such that $\nu_1(E\cap A)=0=\mu(E-A)$ for every $E\in\gamma$, we have $m_1(E\cap A)=m(E\cap A-A)=m(\phi)=0$. Hence $m_1\perp\mu$.

Let m_0' and m_1' be another decomposition for m. i.e $m=m_0'+m_1'$, where $m_0'\ll\mu$ and $m_1'\perp\mu$. Then $m=m_0+m_1=m_0'+m_1'$ and $m_0-m_0'=m_1'-m_1$. Since $m_0\ll\mu$ and $m_0'\ll\mu$, we have $(m_0-m_0')\ll\mu$. Since $m_1\perp\mu$ and $m_1'\perp\mu$, there exist locally measurable set A_1 and A_2 such $m_1(E\cap A_1)=0$, $\mu(E-A_1)=0$, $m_1'(E\cap A_2)=0$ and $\mu(E-A_2)=0$ for every $E\in\gamma$. We put $A=A_1\cap A_2$. Then A is locally measurable. Since $E-A=(E-A_1)\cup(E-A_1)$ we have $\mu(E-A)\leqq\mu(E-A_1)+\mu(E-A_2)=0$. As $E\cap A_2\in\gamma$, we have $m_1(E\cap A)=m_1((E\cap A_2)\cap A_1)=0$ and $\mu(E\cap A_2-A_1)=0$. Similarly $m_1'(E\cap A)=0$ and $\mu(E\cap A_1-A_2)=0$. Hence $(m_1'-m_1)\perp\mu$. Since $(m_0-m_0')\ll\mu$ and $(m_0-m_0')\perp\mu$, by Lemma 1 $m_0=m_0'$ and $m_1=m_1'$.

3. Regular Borel measures

Let S be a locally compact Hausdorff space, $\mathfrak{B}(S)$ ($\mathfrak{B}_0(S)$) the σ-ring generated by the compact (compact G_δ) sets of S and X a normed space.

Definition 4. Any vector measure $m;\ \mathfrak{B}(S)\ (\mathfrak{B}_0(S))\to X$ is called a Borel (Baire) measure on S and any non-negative measure on $\mathfrak{B}(S)$ ($\mathfrak{B}_0(S)$) is called a Borel (Baire) measure if it is a Borel (Baire) measure in the sense of Halmos' (Halmos [6] § 52).

Definition 5. A Borel measure $m;\ \mathfrak{B}(S)\to X$ is called regular if for every $A\in\mathfrak{B}(S)$ and every number $\varepsilon>0$ there exist a compact set $K\subset A$, $K\in\mathfrak{B}(S)$ and an open set $G\supset A$, $G\in\mathfrak{B}(S)$ such that for every $A'\in\mathfrak{B}(S)$ with $A'\subset G-K$ we have $\|m(A')\|<\varepsilon$. The regularity of Baire measure is similar.

We note that the above regularity is equivalent to Halmos' regularity for a finite non-negative measure.

Theorem 2. *Let $m;\ \mathfrak{B}(S)\to X$ be a Borel measure and μ a finite non-negative regular Borel measure. If $m\ll\mu$, then m is regular.*

Proof. Since $m\ll\mu$, by Proposition 1 there exists for every number $\varepsilon>0$ a number $\delta=\delta(\varepsilon)>0$ such that $\mu(A)<\delta\Longrightarrow\|m(A)\|<\varepsilon$. By the regularity of μ, for any $A\in\mathfrak{B}(S)$ and the above number $\delta>0$ there exist a compact set $K\subset A$, $K\in\mathfrak{B}(S)$ and an open set $G\supset A$, $G\in\mathfrak{B}(S)$ such that for every $A'\in\mathfrak{B}(S)$ with $A'\subset G-K$ we have $\mu(A')<\delta$. Hence $\|m(A)\|<\varepsilon$. Therefore m is regular.

Proposition 3. *Let $m;\ \mathfrak{B}(S)\to X$ be a Borel measure and ν the measure determined by Proposition 2. Then m is regular if and only if ν is regular.*

Proof. The necessity is clear by Proposition 2 (2).

The sufficiency is clear by the above theorem.

Proposition 4. *Any Borel measure* $m; \mathfrak{B}(S) \to X$ *is regular if and only if for every compact set* K *there exists a compact* G_δ *set* U *such that* $U \supset K$ *and* $\tilde{m}(U-K)=0$.

Proof. Let ν be the measure determined by Proposition 2.

Necessity. By Proposition 3 ν is regular. Then there exists a sequence $\{U_n\}$ of open Borel sets such that $K \subset U_n$ and $\nu(K)=\inf_n \nu(U_n)$. For each n, by *Halmos* ([6] § 50. Theorem D) there exists a compact G_δ set V_n such that $K \subset V_n \subset U_n$. Then $U=\bigcap_{n=1}^{\infty} V_n$ is a compact G_δ set, $U \supset K$ and $\nu(K) \leqq \nu(U) \leqq \nu(V_n) \leqq \nu(U_n)$ for all n and hence $\nu(K)=\nu(U)$. Since ν is finite, we have $\nu(U-K)=0$. Hence $\tilde{m}(U-K)=0$.

Sufficiency. Since $\nu(E) \leqq \tilde{m}(E)$ for every $E \in \mathfrak{B}(S)$, $\nu(U-K)=0$. Let $\{U_n\}$ be a sequence of open sets such that $U=\bigcap_{n=1}^{\infty} U_n$. By *Halmos* ([6] § 60. Theorem D) there exists, for each n, an open Baire set V_n such that $U \subset V_n \subset U_n$. Then we have $U=\bigcap_{n=1}^{\infty} V_n$. Hence $\lim_{n\to\infty} \nu(\bigcap_{k=1}^{n} V_k)=\nu(U)=\nu(K)$.

Therefore by *Halmos* ([6] § 52. Theorem H) ν is regular. By Proposition 3 m is regular.

Proposition 5. *Any Baire measure* $m; \mathfrak{B}_0(S) \to X$ *is regular.* (*Dinculeanu* and *Kluvanek* [4] Theorem 4).

Proof. By Proposition 2 there exists a finite non-negative measure ν on $\mathfrak{B}_0(S)$ such that $m \ll \nu$. Since ν is finite, ν is a Baire measure. By *Halmos* ([6] § 52. Theorem G) ν is regular. Since $m \ll \nu$, m is regular.

4. The decomposition theorem of Borel measures

Let S be a locally compact Hausdorff space, $\mathfrak{B}(S)$ the σ-ring generated by the compact sets of S and X a normed space.

Definition 6. A Borel measure $m; \mathfrak{B}(S) \to X$ is called antiregular if $m \perp \mu$ for every non-negative regular Borel measure μ.

Lemma 2. *Any Borel measure* $m; \mathfrak{B}(S) \to X$ *which is both regular and antiregular is the zero measure.*

Proof. By Proposition 2 there exists a finite non-negative measure ν on $\mathfrak{B}(S)$ such that $m \ll \nu$. Since m is regular, by Proposition 3 ν is regular. Since m is antiregular $m \perp \nu$. By Lemma 1 m is the zero measure.

Theorem 3. *Let* $m; \mathfrak{B}(S) \to X$ *be a Borel measure. Then there exist unique regular* m_0 *and antiregular* m_1 *such that* $m=m_0+m_1$.

Proof. By Proposition 2 there exists a finite non-negative measure ν on $\mathfrak{B}(S)$ such that $m \ll \nu$. Since ν is finite, ν is a Borel measure. The restriction of ν to $\mathfrak{B}(S)$ is a Baire measure which can be extended to a unique regular Borel measure μ (*Halmos* [6] § 54. Theorem D). By Theorem 1 there exist ν_0 and ν_1 such that $\nu=\nu_0+\nu_1$. where $\nu_0 \ll \mu$ and $\nu_1 \perp \mu$. As the proof of Theorem 1 we define $m_0(E)=m(E \cap A)$ and $m_1(E)=m(E-A)$ for every $E \in \mathfrak{B}(S)$ (where A is some locally measurable set). Since $m_0 \ll \nu_0$ and ν_0 is regular, by Theorem 2 m_0 is regular. By *Johnson* ([2] Theorem 2.3) ν_1 is antiregular and m_1 is also antiregular by $m_1 \ll \nu_1$. Let m_0' be regular and m_1' antiregular with $m=m_0'+m_1'$. Then $m=m_0+m_1=m_0'+m_1'$ and $m_0-m_0'=m_1'-m_1$. Since m_0 and m_0' are regular, also is m_0-m_0'. The antiregularity of $m_1'-m_1$ may be proved along the same way as the uniqueness' proof of Theorem 1. By Lemma 2 we have $m_0=m_0'$ and $m_1=m_1'$.

Proposition 6. *If a Borel measure $m; \mathfrak{B}(S) \rightarrow X$ is antiregular, then there exists a locally Borel set A (i.e. $E \cap A \in \mathfrak{B}(S)$ for every $E \in \mathfrak{B}(S)$) such that $m(E-A)=0$ for every $E \in \mathfrak{B}(S)$ and such that $m(C)=0$ for every compact set $C \subset A$.*

Proof. By Proposition 2 there exists a finite non-negative measure ν on $\mathfrak{B}(S)$ such that $m \ll \nu$. The restriction of ν to $\mathfrak{B}_0(S)$ is a Baire measure which can be extended to a unique regular Borel measure μ. Since m is antiregular, we have $m \perp \mu$. Then there exists a locally Borel set B such that $m(E \cap B)=0$ and $\mu(E-B)=0$ for every $E \in \mathfrak{B}(S)$. If we put $A=B^c$, then A is a locally Borel set and $m(E-A)=0$ for every $E \in \mathfrak{B}(S)$. Now if C is a compact subset of A, evidently $\mu(C)=0$. By regularity of μ there exists a compact G_δ set D such that $D \supset C$ and $\mu(C)=\mu(D)$. Since $0=\mu(C)=\mu(D)=\nu(D) \geqq \nu(C)$, we have $\nu(C)=0$. Hence $m(C)=0$.

Proposition 7. *If $m; \mathfrak{B}(S) \rightarrow X$ is antiregular, then $m(\{s\})=0$ for every point $s \in S$.*

Proof. For each $s \in S$, let v_s be a point measure of s. Then ν_s is a regular Borel measure. Hence $m \perp \nu_s$. It follows that $m(\{s\})=0$.

Proposition 8. *Let $m; \mathfrak{B}(S) \rightarrow X$ be a Borel measure and ν the measure determined by Proposition 2. Then m is antiregular if and only if ν is antiregular.*

Proof. *Necessity.* For every regular non-negative Borel measure μ we have $m \perp \mu$. Then there exists a locally Borel set A such that $m(E \cap A)=0$ and $\mu(E-A)=0$ for every $E \in \mathfrak{B}(S)$. So we have $m(B)=0$ for every $B \in \mathfrak{B}(S)$ with $B \subset E \cap A$. Then by Proposition 2 (2) $\tilde{m}(E \cap A)=0$ implies $\nu(E \cap A)=0$. Therefore ν is antiregular.

The sufficiency is obvious.

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