ON A BILATERAL GENERATING FUNCTION FOR THE ULTRASPHERICAL POLYNOMIALS

By

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1. Introduction:

Recently *Chatterjea* [1] has proved the following bilateral generating function for the ultraspherical polynomials $\{P_n^{(\lambda)}(x)\}$:

If
$$F(x, t)=\sum_{n=0}^{\infty} a_n t^n P_n^{(\lambda)}(x) ,$$

then
$$\rho^{-2\lambda} F\left(\frac{x-t}{\rho}, \frac{ty}{\rho}\right)=\sum_{n=0}^{\infty} t^n b_n(y) P_n^{(\lambda)}(x) ,$$

where
$$b_n(y)=\sum_{r=0}^{n}\binom{n}{r} a_r y^r, \quad \rho=(1-2xt+t^2)^{1/2} ,$$

and
$$P_n^{(\lambda)}(x)=\sum_{m=0}^{[n/2]} \frac{(-1)^m (\lambda)_{n-m}}{m!(n-2m)!}(2x)^{n-2m} .$$

In the present paper we shall point out several applications of *Chatterjea's* formula stated above in deriving various bilateral generating functions for ultraspherical polynomials, some of which are believed to be new. The following bilateral generating functions are the characteristic ones:

(1.2)
$$\sum_{n=0}^{\infty} {}_2F_1(-n, a; b; y) P_n^{(\lambda)}(x) t^n$$
$$=\rho^{-2\lambda} F_1\left(a, \lambda, \lambda; b; \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}), \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right) ,$$

where
$$F_1(a, b, b'; c; x, y)=\sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{n!k!(c)_{n+k}} x^k y^n .$$

(1.3)
$$\sum_{n=0}^{\infty} {}_1F_1(-n; b; y) P_n^{(\lambda)}(x) t^n$$
$$=\rho^{-2\lambda}\phi_2\left(\lambda, \lambda; b; \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}), \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right) ,$$

where
$$\phi_2(\alpha, \beta; \delta; x, y)=\sum_{n,k=0}^{\infty} \frac{(\alpha)_k(\beta)_n}{(\delta)_{n+k} k! n!} x^k y^n .$$

(1.4) $$\sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix}-n, a, b; \\ 2\lambda, \lambda+1/2;\end{matrix} y\right] P_n^{(\lambda)}(x)t^n$$
$$=\rho^{-2\lambda}F_4\left(a, b; \lambda+\frac{1}{2}, \lambda+\frac{1}{2}; \frac{ty}{2\rho^2}(t-x+\rho), \frac{ty}{\rho^2}(t-x-\rho)\right).$$

where F_4 is *Appell's* hypergeometric function of two variables.

2. (A). As *Chatterjea* has pointed out that starting with the formula (1.1) one can derive a large number of bilateral generating functions for the ultraspherical polynomials by attributing different values to a_n we first start with the following generating function due to *Dhawan* [2],

(2.1) $$F_1(a, \lambda, \lambda; b; ut, vt)=\sum_{n=0}^{\infty} \frac{(a)_n P_n^{(\lambda)}(x)}{(b)_n} t^n, \qquad (|t|<1),$$

where $$u=x-\sqrt{x^2-1}, \quad v=x+\sqrt{x^2-1},$$

and $$F_1(a, b, b'; c; x, y)=\sum_{n,k=0}^{\infty} \frac{(a)_{n+k}(b)_n(b')_k}{(c)_{n+k}\, k!\, n!} x^k y^n .$$

Now using $a_n=\dfrac{(a)_n}{(b)_n}$ in (1.1), it follows therefore from (1.1) and (2.1) that

$$\rho^{-2\lambda}F_1\left(a, \lambda, \lambda; b; \frac{yt}{\rho^2}(x-t-\sqrt{x^2-1}), \frac{yt}{\rho^2}(x-t+\sqrt{x^2-1}\right)=\sum_{n=0}^{\infty} b_n(y)t^n P_n^{(\lambda)}(x),$$

where $$b_n(y)=\sum \binom{n}{r} \frac{(a)_r}{(b)_r} y^r = {}_2F_1(-n, a; b; -y).$$

Thus we derive the following bilateral generating function:

(2.2) $$\sum_{n=0}^{\infty} {}_2F_1(-n, a: b; y)P_n^{(\lambda)}(x)t^n$$
$$=\rho^{-2\lambda}F_1\left(a, \lambda, \lambda; b; \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}), \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right),$$

where $$\rho=(1-2xt+t^2)^{1/2}.$$

It is interesting to note some particular cases of our formula (2.2). When we put $b=2\lambda$, we obtain

(2.3) $$\sum_{n=0}^{\infty} {}_2F_1(-n, a; 2\lambda; y)P_n^{(\lambda)}(x)t^n$$
$$=\rho^{-2\lambda}F_1\left(a, \lambda, \lambda; 2\lambda; \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}), \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right).$$

Now we notice that

(2.4) $$F_1(\alpha, \beta, \beta'; \beta+\beta'; x, y)=(1-y)^{-\alpha}{}_2F_1\left(\alpha, \beta; \beta+\beta'; \frac{x-y}{1-y}\right).$$

(2.5) $${}_2F_1(a, b; 2b; z)=\left(1-\frac{z}{2}\right)^{-a}{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; b+\frac{1}{2}; \frac{z^2}{(2-z)^2}\right).$$

It follows therefore from (2.4) and (2.5) that

$$F_1\left(a, \lambda, \lambda; 2\lambda; \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}), \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right)$$
$$=\rho^{2a}[\rho^2+yt(x-t)]^{-a}\times{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; \lambda+\frac{1}{2}; \frac{y^2t^2(x^2-1)}{\{\rho^2+yt(x-t)\}^2}\right).$$

Thus we have

(2.6) $$\sum_{n=0}^{\infty}{}_2F_1(-n, a; 2\lambda; y)P_n^{(\lambda)}(x)t^n$$
$$=\rho^{2(a-\lambda)}[\rho^2+yt(x-t)]^{-a}{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; \lambda+\frac{1}{2}; \frac{y^2t^2(x^2-1)}{\{\rho^2+yt(x-t)\}^2}\right).$$

Here we may add that the result (2.6) has been derived by *Chatterjea* [1] from (1.1) and the following well-known generating function of *Brafman*:

(2.7) $$(1-xt)^{-a}{}_2F_1\left(\frac{a}{2}, \frac{a+1}{2}; \lambda+\frac{1}{2}; \frac{t^2(x^2-1)}{(1-xt)^2}\right)=\sum_{m=0}^{\infty}\frac{(a)_m t^m}{(2\lambda)_m}P_m^{(\lambda)}(x).$$

Thus our result (2.2) may be considered as a generalization of (2.6).

2. (B). Secondly we note [2, p. 92] the the following generating function

(2.8) $$\phi_2(\lambda, \lambda; b; ut, vt)=\sum_{n=0}^{\infty}\frac{P_n^{(\lambda)}(x)}{(b)_n}t^n, \quad (|t|<1);$$

where $$u=x-\sqrt{x^2-1}, \quad v=x+\sqrt{x^2-1},$$

and $$\phi_2(\alpha, \beta; \delta; x, y)=\sum_{n,k=0}^{\infty}\frac{(\alpha)_k(\beta)_n x^k y^n}{(\delta)_{n+k}\, n!\, k!}.$$

Now using $a_n=\dfrac{1}{(b)_n}$ in *Chatterjea's* formula, we obtain by means of (2.8):

$$\sum_{n=0}^{\infty}b_n(y)P_n^{(\lambda)}(x)t^n=\rho^{-2\lambda}\phi_2\left(\lambda, \lambda; b; \frac{yt}{\rho^2}(x-t-\sqrt{x^2-1}), \frac{yt}{\rho^2}(x-t+\sqrt{x^2-1})\right),$$

where $$b_n(y)=\sum_{k=0}^{n}\binom{n}{k}\frac{y^k}{(b)_n}={}_1F_1(-n; b; -y).$$

Thus we derive the following bilateral generating function:

(2.9) $$\sum_{n=0}^{\infty} {}_1F_1(-n;\, b;\, y) P_n^{(\lambda)}(x) t^n$$

$$=\rho^{-2\lambda}\phi_2\left(\lambda,\, \lambda;\, b;\, \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}),\, \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right).$$

Since we know that

$$L_n^{(\alpha)}(x)=\frac{(1+\alpha)_n}{n!}\, {}_1F_1(-n;\, 1+\alpha;\, x)\, ,$$

We obtain from (2.9)

(2.10) $$\sum_{n=0}^{\infty} \frac{n!}{(b)_n} L_n^{(b-1)}(y) P_n^{(\lambda)}(x) t^n$$

$$=\rho^{-2\lambda}\phi_2\left(\lambda,\, \lambda;\, b;\, \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}),\, \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right).$$

In particular, when $b=2\lambda$, we derive from (2.10)

(2.11) $$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} L_n^{(2\lambda-1)}(y) P_n^{(\lambda)}(x) t^n$$

$$=\rho^{-2\lambda}\phi_2\left(\lambda,\, \lambda;\, 2\lambda;\, \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}),\, \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right),$$

which can be compared with the following generating function of *Weisner* [3]:

(2.12) $$\sum_{n=0}^{\infty} \frac{n!}{(2\lambda)_n} L_n^{(2\lambda-1)}(y) P_n^{(\lambda)}(x) t^n$$

$$=\rho^{-2\lambda}\cdot\exp\left\{\frac{-yt(x-t)}{\rho^2}\right\}\cdot {}_0F_1\left(-;\, \lambda+\frac{1}{2};\, \frac{y^2t^2(x^2-1)}{4\rho^4}\right).$$

An easy comparison of (2.11) and (2.12) leads us to the following result:

(2.13) $$\phi_2\left(\lambda,\, \lambda;\, 2\lambda;\, \frac{yt}{\rho^2}(t-x+\sqrt{x^2-1}),\, \frac{yt}{\rho^2}(t-x-\sqrt{x^2-1})\right)$$

$$=\exp\left\{\frac{-yt(x-t)}{\rho^2}\right\}\cdot {}_0F_1\left(-;\, \lambda+\frac{1}{2};\, \frac{y^2t^2(x^2-1)}{4\rho^4}\right).$$

Thus our result (2.10) can be considered as a generalization of *Weisner's* result viz. (2.12).

2. (C). Lastly we take the following generating function due to *Brafman*:

(2.14) $$\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(\lambda+1/2)_n(2\lambda)_n} P_n^{(\lambda)}(x) t^n = F_4\left(a,\, b;\, \lambda+\frac{1}{2},\, \lambda+\frac{1}{2};\, \frac{1}{2}t(x-1),\, \frac{1}{2}t(x+1)\right),$$

where F_4 is *Appell's* hypergeometric function of two variables.

Putting $a_n = \dfrac{(a)_n(b)_n}{(2\lambda)_n(\lambda+1/2)_n}$ in *Chatterjea's* formula we obtain by means of (2.14):

$$\sum_{n=0}^{\infty} {}_3F_2\left[\begin{matrix} -n, a, b; \\ 2\lambda, \lambda+1/2; \end{matrix}\; y\right] P_n^{(\lambda)}(x) t^n \tag{2.15}$$

$$= \rho^{-2\lambda} F_4\left(a, b; \lambda+\frac{1}{2}, \lambda+\frac{1}{2}; \frac{ty}{2\rho^2}(t-x+\rho), \frac{ty}{2\rho^2}(t-x-\rho)\right).$$

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