A SINGULAR INTEGRAL EQUATION WITH A GENERALIZED MITTAG LEFFLER FUNCTION IN THE KERNEL

By

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1. Introduction.

In recent years several authors ([1], [3], [8], [11], [13]) applied the Laplace transform to solve convolution equations which are special cases of

$$(1.1) \qquad \int_0^x \frac{(x-t)^{b-1}}{\Gamma(b)} {}_1F_1(a; b; c(x-t)) f(t)dt \overset{\circ}{=} g(x) \qquad \mathrm{Re}\, b>0\,,$$

discussed by the present author [7] by a use of fractional integration. The purpose of this paper is to discuss an integral equation of a much more general nature, viz. equation

$$(1.2) \qquad \int_a^x (x-t)^{\beta-1} E^{\rho}_{\alpha,\beta} \lambda(x-t)^{\alpha} f(t)dt \overset{\circ}{=} g(x) \qquad \mathrm{Re}\, \beta>0\,,$$

for any real number $a \geqslant 0$ where the function

$$(1.3) \qquad E^{\rho}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\rho)_n z^n}{\Gamma(\alpha n+\beta) n!} \qquad \mathrm{Re}\, \alpha>0\,,$$

is an entire function of order $(\mathrm{Re}\, \alpha)^{-1}$ and contains several well-known special functions as particular cases.

We define a linear operator $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$ on a space L of functions by the integral in (1.2) and employ an operator of fractional integration $I^{\mu}: L \to L$ to prove results on $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$; these results are subsequently used to discuss theorems on the solutions of (1.2). The technique used can be applied to obtain analogous results on the integral equation

$$(1.4) \qquad \mathfrak{E}^*(\alpha, \beta; \rho; \lambda) f(x) \equiv \int_x^b (t-x)^{\beta-1} E^{\rho}_{\alpha,\beta} \lambda(t-x)^{\alpha} f(t)dt \overset{\circ}{=} g(x) \qquad \mathrm{Re}\, \beta>0\,,$$

which contains as particular cases the equations considered in [9] and [10]. The symbol $\overset{\circ}{=}$ is the usual notation for equality a.e.

2. Definitions and preliminary results.

The function $E^{\rho}_{\alpha,\beta}(z)$. We define $E^{\rho}_{\alpha,\beta}(z)$ by the series (1.3), the parameters

α, β and ρ being complex numbers with $\mathrm{Re}\,\alpha>0$. Some of the well-known functions which are particular cases of $E^{\rho}_{\alpha,\beta}(z)$ are the Mittag Leffler's function $E_{\alpha}(z)$ ((5), (6)), the *Wiman*'s function $E_{\alpha,\beta}(z)$ [12] and the confluent hypergeometric function ${}_1F_1(\rho;\beta;z)$. Indeed

$$E_{\alpha}(z)=E^{1}_{\alpha,1}(z),\ E_{\alpha,\beta}(z)=E^{1}_{\alpha,\beta}(z)\ \text{and}\ {}_1F_1(\rho;\beta;z)=\Gamma(\beta)\cdot E^{\rho}_{1,\beta}(z)\,.$$

When α is a positive integer, say n, then

$$E^{\rho}_{n,\beta}(z)=\frac{1}{\Gamma(\beta)}\,{}_1F_n\left(\rho;\frac{\beta}{n},\frac{\beta+1}{n},\dots,\frac{\beta+n-1}{n};\frac{z}{n^n}\right);$$

also

$$h_i(x,n)=x^{i-1}E^{1}_{n,i}(x^n)\,,$$

$$k_i(x,n)=x^{i-1}E^{1}_{n,i}(-x^n)\,,$$

h_i and k_i being generalized hyperbolic and generalized trigonometrical functions [2]. If ρ is a negative integer say $-n$ and α is a positive integer k, then one set of the biorthogonal polynomial pair discussed by *Konhauser* [4] is given by

$$Z^{c}_{n}(x;k)=\Gamma(kn+c+1)E^{-n}_{k,c+1}(x^k)\,.$$

The polynomial $Z^{c}_{n}(x;k)$ is related to $E^{\rho}_{\alpha,\beta}(z)$ just the same way as the Laguerre polynomial $L^{\alpha}_{n}(x)$ is related to *Kummer*'s ${}_1F_1$.

The function $E^{\rho}_{\alpha,\beta}(z)$ as well as the polynomial $Z^{c}_{n}(x;k)$ has a number of properties which may be of independent interest. We give below a few results which can be easily verified:—

(2.1) $$\left(\frac{d}{dz}\right)^{m}E^{\rho}_{\alpha,\beta}(z)=(\rho)_m E^{\rho+m}_{\alpha,\beta+m\alpha}(z)\,.$$

(2.2) $$\left(\frac{d}{dz}\right)^{m}[z^{\beta-1}E^{\rho}_{\alpha,\beta}(z^{\alpha})]=z^{\beta-m-1}E^{\rho}_{\alpha,\beta-m}(z^{\alpha})\,.$$

(2.3) $$\left(z\frac{d}{dz}+\rho\right)E^{\rho}_{\alpha,\beta}(z)=\rho E^{\rho+1}_{\alpha,\beta}(z)\,,$$

(2.4) $$(\beta-\alpha\rho-1)E^{\rho}_{\alpha,\beta}(z)=E^{\rho}_{\alpha,\beta-1}(z)-\alpha\rho E^{\rho+1}_{\alpha,\beta}(z)\,,$$

(2.5) $$\mathfrak{L}\{t^{\beta-1}E^{\rho}_{\alpha,\beta}(\lambda t^{\alpha})\}=p^{-\beta}(1-\lambda p^{-\alpha})^{-\rho}\ \text{for}\ \mathrm{Re}\,\beta,\ \mathrm{Re}\,p>0,\ |p|>|\lambda|^{\frac{1}{\mathrm{Re}\,\alpha}}\,,$$

where $\mathfrak{L}\{f(t)\}$ denotes the Laplace transform of $f(t)$.

The operator I^{μ}. L denotes the linear space of (equivalent classes of) complex-valued functions f which are L-integrable on a finite $[a,b]$, $a\geqslant 0$ with the norm

$\|f\|=\int_a^b |f(t)|dt$. For complex μ with $\mathrm{Re}\,\mu>0$, $I^\mu: L\rightarrow L$ is a linear operator defined by the fractional integral

(2.6) $$I^\mu f(x)=\int_a^x \frac{(x-t)^{\mu-1}}{\Gamma(\mu)}f(t)dt \quad \text{for almost all } x\varepsilon(a, b)\,.$$

It is easily verified that I^μ is bounded and it is a standard result that $I^\mu f=0 \Longrightarrow f=0$, so that the inverse operator exists on subspace L_μ of L. If $0<\mathrm{Re}\,\mu<\mathrm{Re}\,\nu$, then it is easily proved that $L_\nu\subset L_\mu\subset L$ and the inclusion is proper.

For $\mathrm{Re}\,\mu<0$, I^μ is defined as the inverse of $I^{-\mu}$. If $\mathrm{Re}\,\mu\neq 0$, $\mathrm{Re}\,\nu\neq 0$, then $I^\mu I^\nu f=I^{\mu+\nu}f$ for suitable functions f. For $\mathrm{Re}\,\mu=0$, I^μ is defined on L_λ with $\mathrm{Re}\,\lambda>0$ as $I^{-1}I^{1+\mu}$.

Theorem 1. *If* $\mathrm{Re}\,\mu>0$ *and* $f\varepsilon L$, *then the integral*

(2.7) $$\int_a^x (x-t)^{\mu-1}E^\rho_{\alpha,\beta}\lambda(x-t)^\alpha f(t)dt$$

defines a function in L.

It is sufficient to prove that

(2.8) $$\int_a^b dx\int_a^x |(x-t)^{\mu-1}E^\rho_{\alpha,\beta}\lambda(x-t)^\alpha f(t)|dt<\infty\,.$$

The integral in (2.8) is at most

$$\int_a^b |f(t)|dt\int_t^b |(x-t)^{\mu-1}E^\rho_{\alpha,\beta}\lambda(x-t)^\alpha|dx\leqslant\int_a^b |f(t)|dt\int_0^b |v^{\mu-1}E^\rho_{\alpha,\beta}\lambda v^\alpha|dv\,.$$

The entire function $E^\rho_{\alpha,\beta}(z)$ is bounded in $[a, b]$, let

$$|E^\rho_{\alpha,\beta}\lambda v^\alpha|\leqslant M \qquad \text{for } v\varepsilon[a, b]\,.$$

Hence the double integral does not exceed

$$M(\mathrm{Re}\,\mu)^{-1}b^{\mathrm{Re}\,\mu}\|f\|\,.$$

The operator $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$. For complex $\alpha, \beta, \rho, \lambda$ with $\mathrm{Re}\,\beta>0$ and $f\varepsilon L$, the linear operator $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$ on L into itself is defined by

$$\mathfrak{E}(\alpha, \beta; \rho; \lambda)f(x)=\int_a^x (x-t)^{\beta-1}E^\rho_{\alpha,\beta}\lambda(x-t)^\alpha f(t)dt \qquad a<x<b\,.$$

For brevity we shall denete $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$ by $\mathfrak{E}(\beta)$ when it is understood that the other parameters are unaltered.

Theorem 2. *If* $\mathrm{Re}\,\mu>0$, $\mathrm{Re}\,\beta>0$, *then*

(2.9) $$\frac{1}{\Gamma(\mu)}\int_t^x (x-s)^{\mu-1}(s-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(s-t)^{\alpha}ds=(x-t)^{\beta+\mu-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}\ ,$$

(2.10) $$\frac{1}{\Gamma(\mu)}\int_t^x (s-t)^{\mu-1}(x-s)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-s)^{\alpha}ds=(x-t)^{\beta+\mu-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}\ .$$

For Re μ, Re $\beta>0$ and z any complex number

(2.11) $$\frac{1}{\Gamma(\mu)}\int_0^1 (1-u)^{\mu-1}u^{\beta-1}E^{\rho}_{\alpha,\beta}(zu^{\alpha})du$$
$$=\sum_{n=0}^{\infty}\frac{(\rho)_n z^n}{\Gamma(\mu)\Gamma(\alpha n+\beta)n!}\int_0^1 (1-u)^{\mu-1}u^{\alpha n+\beta-1}du=E^{\rho}_{\alpha,\beta+\mu}(z)\ ,$$

interchanging the order of integration and summation and evaluating the Eulerian integral. The justification for the interchange is provided by Lebesgue's theorem. Finally (2.9) is obtained by putting in (2.11), $u=\dfrac{s-t}{x-t}$, $z=\lambda(x-t)^{\alpha}$; to prove (2.10) put $u=\dfrac{x-s}{x-t}$, $z=\lambda(x-t)^{\alpha}$.

3. Results on the operator $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$.

Theorem 3. *If* Re $\mu>-$Re β, *then operating on* L

(3.1) $$I^{\mu}\mathfrak{E}(\alpha, \beta; \rho; \lambda)=\mathfrak{E}(\alpha, \beta+\mu; \rho; \lambda)\ ,$$

that is, $$I^{\mu}\mathfrak{E}(\beta)=\mathfrak{E}(\beta+\mu)\ .$$

(i) Suppose Re $\mu>0$. For $f\varepsilon L$, $\mathfrak{E}(\beta)f$ is in L so that $I^{\mu}\mathfrak{E}(\beta)f(x)$ exists in L and equals

$$\frac{1}{\Gamma(\mu)}\int_a^x (x-s)^{\mu-1}ds\int_a^s (s-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(s-t)^{\alpha}f(t)dt$$
$$\overset{\circ}{=}\frac{1}{\Gamma(\mu)}\int_a^x f(t)dt\int_t^x (x-s)^{\mu-1}(s-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}ds$$
$$=\int_a^x (x-t)^{\beta+\mu-1}E^{\rho}_{\alpha,\beta+\mu}\lambda(x-t)^{\alpha}f(t)dt \qquad \text{by (2.9)}\ .$$

The correctness of the inversion in the order of integration is easily verified by an application of *Fubini's* theorem.

(ii) Suppose $0>$Re $\mu>-$Re β. Since Re $(\beta+\mu)>0$, for f in L, $\mathfrak{E}(\beta+\mu)f$ exists in L. Also by (i), $I^{-\mu}\mathfrak{E}(\beta+\mu)f=\mathfrak{E}(\beta)f$. As $\mathfrak{E}(\beta+\mu)f$ exists in L, we can write $\mathfrak{E}(\beta+\mu)f=I^{\mu}\mathfrak{E}(\beta)f$.

(iii) Let Re $\mu=0$. By case (ii)

$$\mathfrak{E}(\beta+\mu)=I^{\mu-1}\mathfrak{E}(\beta+1)=I^{\mu}[I^{-1}\mathfrak{E}(\beta+1)]=I^{\mu}\mathfrak{E}(\beta)\ .$$

Theorem 4. *If* Re $\mu>0$ *and* $f\varepsilon L$, *then for almost all* $x\varepsilon(a, b)$

$$I^{\mu}\mathfrak{E}(\beta)f(x)=\mathfrak{E}(\beta)I^{\mu}f(x) . \tag{3.2}$$

Since Re $\mu>0$, $I^{\mu}f$ is in L and

$$\begin{aligned}\mathfrak{E}(\beta)I^{\mu}f(x)&=\int_a^x(x-u)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-u)^{\alpha}du\int_a^u\frac{(u-t)^{\mu-1}}{\Gamma(\mu)}f(t)dt\\&\overset{\circ}{=}\frac{1}{\Gamma(\mu)}\int_a^x f(t)dt\int_t^x(x-u)^{\beta-1}(u-t)^{\mu-1}E^{\rho}_{\alpha,\beta}\lambda(x-u)^{\alpha}du\\&=\int_a^x(x-t)^{\beta+\mu-1}E^{\rho}_{\alpha,\beta+\mu}\lambda(x-t)^{\alpha}f(t)dt \qquad \text{using (2.10)}\\&=\mathfrak{E}(\beta+\mu)f(x)=I^{\mu}\mathfrak{E}(\beta)f(x) , \qquad \text{by Theorem 3.}\end{aligned}$$

Theorem 4a. *If* Re $\mu\leqslant 0$ *and* $I^{\mu}f$ *exists in* L, *then also* (3.2) *holds.*

(i) Suppose Re $\mu<0$, let $I^{\mu}f(x)=\phi(x)$. By Theorem 4

$$I^{-\mu}\mathfrak{E}(\beta)\phi(x)=\mathfrak{E}(\beta)I^{-\mu}\phi(x) .$$

But $\mathfrak{E}(\beta)\phi(x)$ exists in L, so that

$$\mathfrak{E}(\beta)\phi(x)=I^{\mu}\mathfrak{E}(\beta)I^{-\mu}\phi(x) ,$$

that is,

$$\mathfrak{E}(\beta)I^{\mu}f(x)=I^{\mu}\mathfrak{E}(\beta)f(x) .$$

(ii) When Re $\mu=0$, we write $I^{\mu+1}\mathfrak{E}(\beta)f(x)=\mathfrak{E}(\beta)I^{\mu+1}f(x)$

i.e.

$$I^{\mu}\mathfrak{E}(\beta)f(x)=I^{-1}\mathfrak{E}(\beta)I^{\mu+1}f(x)=\mathfrak{E}(\beta)I^{-1}[I^{\mu+1}f(x)] .$$

Theorems 4 and 4a together can be combined in

Theorem 4b. *If* f *and* $I^{\mu}f$ *exist in* L, μ *being any complex number, then*

$$I^{\mu}\mathfrak{E}(\beta)f(x)=\mathfrak{E}(\beta)I^{\mu}f(x) ,$$

that is, the operator $\mathfrak{E}(\beta)$ *commutes with* I^{μ}.

Theorem 5. *For* Re β, Re $\beta'>0$, *operating on* L

$$\mathfrak{E}(\alpha, \beta; \rho; \lambda)\mathfrak{E}(\alpha, \beta'; \rho'; \lambda)=\mathfrak{E}(\alpha, \beta+\beta'; \rho+\rho'; \lambda) . \tag{3.3}$$

For $f\varepsilon L$ and $x\varepsilon(a, b)$

$$\begin{aligned}&\mathfrak{E}(\alpha, \beta; \rho; \lambda)\mathfrak{E}(\alpha, \beta'; \rho'; \lambda)f(x)\\&=\int_a^x(x-u)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-u)^{\alpha}du\int_a^u(u-t)^{\beta'-1}E^{\rho'}_{\alpha,\beta'}(u-t)^{\alpha}f(t)dt\\&\overset{\circ}{=}\int_a^x f(t)dt\int_t^x(x-u)^{\beta-1}(u-t)^{\beta'-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}E^{\rho'}_{\alpha,\beta'}\lambda(u-t)^{\alpha}du ,\end{aligned} \tag{3.4}$$

reversing the order of integration which is easily justified.

Putting $v=\dfrac{x-u}{x-t}$, the inner integral is

$$(3.5)\qquad (x-t)^{\beta+\beta'-1}\int_0^1 v^{\beta-1}(1-v)^{\beta'-1}E^{\rho}_{\alpha,\beta}\{\lambda(x-t)^{\alpha}v^{\alpha}\}E^{\rho'}_{\alpha,\beta'}\{\lambda(x-t)^{\alpha}(1-v)^{\alpha}\}dv$$

$$=(x-t)^{\beta+\beta'-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{(\rho)_m(\rho')_n\lambda^{m+n}(x-t)^{\alpha(m+n)}}{\Gamma(\alpha m+\beta)\Gamma(\alpha n+\beta')m!n!}\int_0^1 v^{\alpha m+\beta-1}(1-v)^{\alpha n+\beta'-1}dv$$

$$=(x-t)^{\beta+\beta'-1}\sum_{m=0}^{\infty}\frac{\lambda^m(x-t)^{\alpha m}}{\Gamma(\alpha m+\beta+\beta')}\sum_{n=0}^{m}\frac{(\rho)_{m-n}(\rho')_n}{(m-n)!n!}$$

$$=(x-t)^{\beta+\beta'-1}E^{\rho+\rho'}_{\alpha,\beta+\beta'}\lambda(x-t)^{\alpha}\ .$$

The change in the order of integration and summation in (3.5) is not difficult to justify.

Theorem 6. *For* Re $\beta>0$ *and* $f\varepsilon L$,

$$(3.6)\qquad I^{-\beta}\mathfrak{E}(\alpha,\beta;\rho;\lambda)f(x)\overset{\circ}{=}f(x)+\alpha\rho\lambda\int_a^x (x-t)^{\alpha-1}E^{\rho+1}_{\alpha,\alpha+1}\lambda(x-t)^{\alpha}f(t)dt\ .$$

By Theorem 3

$$I^{1-\beta}\mathfrak{E}(\alpha,\beta;\rho;\lambda)f(x)\overset{\circ}{=}\mathfrak{E}(\alpha,1;\rho;\lambda)f(x)=\int_a^x E^{\rho}_{\alpha,1}\lambda(x-t)^{\alpha}f(t)dt\ ,$$

so that

$$(3.7)\qquad \mathfrak{E}(\alpha,\beta;\rho;\lambda)f(x)=I^{\beta}\frac{d}{dx}\int_a^x E^{\rho}_{\alpha,1}\lambda(x-t)^{\alpha}f(t)dt$$

$$=I^{\beta}\left[f(x)+\alpha\rho\lambda\int_a^x (x-t)^{\alpha-1}E^{\rho+1}_{\alpha,\alpha+1}\lambda(x-t)^{\alpha}f(t)dt\right],$$

which at once gives the desired result.

4. The integral equation (1.2).

We apply the results of the previous section to solve in Theorem 8, the integral equation (1.2) under a condition which is slightly more restrictive than the necessary condition of

Theorem 7. *The existence of* $I^{-\beta}g$ *in* L *is a necessary condition for the integral equation*

$$(4.1)\qquad \int_a^x (x-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}f(t)dt\overset{\circ}{=}g(x)\ ,$$

to admit a solution f *in* L.

Suppose (4.1) has a solution $f\varepsilon L$. From (3.7), the equation can be written as

(4.2) $$I^{\beta}\left[f(x)+\alpha\rho\lambda\int_a^x (x-t)^{\alpha-1}E^{\rho+1}_{\alpha,\alpha+1}\lambda(x-t)^{\alpha}f(t)dt\right]\overset{\circ}{=}g(x)\ .$$

For $f\varepsilon L$, the integral in (4.2) is easily seen by Theorem 1 to exist in L, since $\mathrm{Re}\,\alpha>0$. Consequently $I^{-\beta}g$ exists in L.

Theorem 8. *If* $\mathrm{Re}\,\gamma>\mathrm{Re}\,\beta>0$ *and* $I^{-\gamma}g$ *exists in* L, *then the integral equation*

(4.3) $$\int_a^x (x-t)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(x-t)^{\alpha}f(t)dt\overset{\circ}{=}g(x)\ ,$$

for $a<x<b$, *possesses a solution* (*rather a class of equivalent solutions*) f *in* L *given by*

(4.4) $$f(x)=\int_a^x (x-t)^{\gamma-\beta-1}E^{-\rho}_{\alpha,\gamma-\beta}\lambda(x-t)^{\alpha}I^{-\gamma}g(t)dt\ .$$

In our operator notation, (4.3) and (4.4) are respectively

(4.5) $$\mathfrak{E}(\alpha,\beta;\rho;\lambda)f(x)\overset{\circ}{=}g(x)\ ,$$

(4.6) $$\mathfrak{E}(\alpha,\gamma-\beta;-\rho;\lambda)I^{-\gamma}g(x)=f(x)\ .$$

Substituting for $f(x)$ from (4.6), the left hand member of (4.5) becomes

(4.7) $$\begin{aligned}&\mathfrak{E}(\alpha,\beta;\rho;\lambda)\mathfrak{E}(\alpha,\gamma-\beta;-\rho;\lambda)I^{-\gamma}g(x)\\&=\mathfrak{E}(\alpha,\gamma;0;\lambda)I^{-\gamma}g(x)\qquad\text{by Theorem 5}\\&=g(x)\ ,\end{aligned}$$

since it is easily verified that $\mathfrak{E}(\alpha,\gamma;0;\lambda)\phi(x)=I^{\gamma}\phi(x)$.

Corollary 8.1. *Under the conditions of the above theorem* (4.3) *and* (4.4) *imply each other.*

It is enough to show that

$$\mathfrak{E}(\alpha,\gamma-\beta;-\rho;\lambda)I^{-\gamma}\mathfrak{E}(\alpha,\beta;\rho;\lambda)f(x)=f(x)\ .$$

But by Theorem 4 the left hand member is

$$\mathfrak{E}(\alpha,\gamma-\beta;-\rho;\lambda)\mathfrak{E}(\alpha,\beta;\rho;\lambda)I^{-\gamma}f(x)\ ,$$

so that the results follows as in (4.7).

Remark 1. When $\alpha=1$, $a=0$ and γ is a positive integer we get the transform pair by Wimp (13) obtained by the use of the Laplace transform.

Remark 2. For $a=0$ and positive integral values of γ, Corollary 8.1 can also be proved by the method of the Laplace transform, using (2.5).

5. The integral equation (1.4).

This integral equation can be discussed by a use of the fractional integration operator J^{μ} defined by

$$J^{\mu}f(x)=\int_{x}^{b}\frac{(t-x)^{\mu-1}}{\Gamma(\mu)}f(t)dt\,. \tag{5.1}$$

In fact it can be verified that all the results analogous to theorems 2–6 hold for $\mathfrak{E}^*(\alpha, \beta; \rho; \gamma)$. Plainly J^{μ} plays the same role in this discussion as I^{μ} does for that of $\mathfrak{E}(\alpha, \beta; \rho; \lambda)$. The existence of $J^{-\beta}g$ is a necessary whereas the existence of $J^{-\gamma}g$ for $\operatorname{Re}\gamma>\operatorname{Re}\beta$ is a sufficient condition for (1.4) to admit a unique solution. Corresponding to Corollary 8.1 we have

Theorem 9. *If* $\operatorname{Re}\gamma>\operatorname{Re}\beta>0$, $f\varepsilon L$ *and* $J^{-\gamma}g$ *exists in* L, *then*

$$\int_{x}^{b}(t-x)^{\beta-1}E^{\rho}_{\alpha,\beta}\lambda(t-x)^{\alpha}f(t)dt\overset{\circ}{=}g(x)\,, \tag{5.2}$$

$$\int_{x}^{b}(t-x)^{\gamma-\beta-1}E^{-\rho}_{\alpha,\gamma-\beta}\lambda(t-x)^{\alpha}J^{-\gamma}g(t)dt\overset{\circ}{=}f(x)\,, \tag{5.3}$$

imply each other.

For $\alpha=1$, and by further specialization of parameters, (5.2) reduces to the integral equations solved by Saxena ([9], [10]).

Remark. The results can be extended to the case $b=\infty$ provided the functions f and g are suitably restricted.

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