A NOTE ON MILDLY PARACOMPACT SPACES

By

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A space X is said to be mildly-paracompact if every countable, regular open covering (that is a covering consisting of regularly open sets) of X has a locally-finite open refinement. This concept has been introduced and studied in [3]. In the present note, we propose to obtain some characterisations of mildly-paracompact spaces similar to the characterisations of countably-paracompact spaces obtained in [2].

Definitions. Let $\mathscr{A}$ and $\mathscr{B}$ be two families of subsets of a space X. Then $\mathscr{A}$ is said to be *linearly cushioned* in $\mathscr{B}$ with cushion map $f: \mathscr{A} \rightarrow \mathscr{B}$ if there is a linear ordering '$<$' on $\mathscr{A}$ such that for every subfamily $\mathscr{A}'$ of $\mathscr{A}$ for which there exists an $A \in \mathscr{A}$ such that $A' < A$ for all $A' \in \mathscr{A}'$ we have

$$\overline{\cup\{A' : A' \in \mathscr{A}'\}} \subseteq \cup\{f(A') : A' \in \mathscr{A}'\} .$$

$\mathscr{A}$ is said to be *order cushioned* in $\mathscr{B}$ with cushion map $f: \mathscr{A} \rightarrow \mathscr{B}$ if there is a well ordering '$<$' on $\mathscr{A}$ such that for every subfamily $\mathscr{A}'$ of $\mathscr{A}$ and an $A \in \mathscr{A}$ such that $A' < A$ for all $A' \in \mathscr{A}'$, we have

$$Cl_A[\cup\{A' \cap A : A' \in \mathscr{A}'\}] \subseteq \cup\{f(A') : A' \in \mathscr{A}'\} .$$

The above definition of linearly-cushioned is due to J. E. Vaughan [6] and of order-cushioned has been discussed in [4]. The definition of linearly-cushioned with respect to a well ordering is due to *H. Tamano* [5].

We shall now give some definitions due to *J.R. Boone* [1]. A family $\mathscr{A}$ of subsets of a space is said to be *compact-finite* (resp. *cs-finite*) if every compact set (resp. every set which is closure of a convergent sequence) intersects finitely many members of $\mathscr{A}$. $\mathscr{A}$ is said to be *strongly compact-finite* (resp. *strongly cs-finite*) if the family of closures of members of $\mathscr{A}$ is compact-finite (resp. cs-finite). A family $\mathscr{K}$ of subsets of a space X is said to be an *F-hereditary-collection* if it is a covering of X and if for every closed subset F of X, $F \cap K \in \mathscr{K}$ for all $K \in \mathscr{K}$. A family $\mathscr{A}$ of subsets of X is said to be *$\mathscr{K}$-finite* if every member of $\mathscr{K}$ intersects finitely many members of $\mathscr{A}$. A space X is said to have *W-weak topology* with respect to an F-hereditary collection $\mathscr{K}$ if a subset U of X is open iff $U \cap K$ is open in K for each $K \in \mathscr{K}$.

Theorem 1. *For a normal space X, the following are equivalent*:

(a) *X is mildly-paracompact.*

(b) *Every countable, regular open covering of X has an open cushioned refinement.*

(c) *Every countable, regular open covering of X has a σ-cushioned open refinement.*

(d) *Every countable, regular open covering of X has a linearly-cushioned, open refinement.*

(e) *Every countable, regular open covering of X has an order cushioned open refinement.*

Proof. (a) $\Longrightarrow$ (c). Let $\mathscr{G}=\{G_i : i\in N\}$ be a countable regular open covering of X. Since X is mildly-paracompact, therefore there exists a locally-finite open refinement $\mathscr{H}=\{H_\alpha : \alpha\in\Lambda\}$ of $\mathscr{G}$. Since X is normal, $\mathscr{H}$ is shrinkable, that is, there exists another locally-finite open covering $\mathscr{H}^*=\{H_\alpha^* : \alpha\in\Lambda\}$ of X such that $\overline{H_\alpha^*}\subseteq H_\alpha$ for all $\alpha\in\Lambda$. It is easy to verify now, that $\mathscr{H}^*$ is an open cushioned refinement of $\mathscr{G}$.

(b) $\Longrightarrow$ (c) is obvious.

(c) $\Longrightarrow$ (d). It is easy to verify that every σ-cushioned refinement is linearly-cushioned. Hence the implication.

(d) $\Longrightarrow$ (e). Let $\mathscr{G}$ be any countable, regular open covering of X. There exists a linearly-cushioned, open refinement $\mathscr{H}$ of $\mathscr{G}$ with cushion map $f:\mathscr{H}\to\mathscr{G}$. It may be assumed without any loss of generality that the ordering on $\mathscr{H}$ is a well ordering (cf. [6], Theorem 1). Let $\mathscr{H}'$ be any subfamily of $\mathscr{H}$ and let $H\in\mathscr{H}$ such that $H'<H$ for all $H'\in\mathscr{H}'$. Then, $Cl_H[\cup\{H'\cap H : H'\in\mathscr{H}'\}]=Cl_H[(\cup\{H' : H'\in\mathscr{H}'\})\cap H]=\overline{\cup\{H' : H'\in\mathscr{H}'\}}\cap H\subseteq\overline{\cup\{H' : H'\in\mathscr{H}'\}}\cap H\subseteq\overline{\cup\{H' : H'\in\mathscr{H}'\}}\subseteq\cup\{f(H') : H'\in\mathscr{H}'\}$. This shows that $\mathscr{H}$ is order cushioned in $\mathscr{G}$.

(e) $\Longrightarrow$ (a). Let $\mathscr{A}$ be any countable, regular open covering of X. Let $\mathscr{B}$ be an open, order cushioned refinement of $\mathscr{A}$ with cushion map $f:\mathscr{B}\to\mathscr{A}$. We shall construct a cushioned refinement of $\mathscr{A}$. Let $C_B=B\sim\cup\{B' : B'<B\}$ for each $B\in\mathscr{B}$. Let $\mathscr{C}=\{C_B : B\in\mathscr{B}\}$. Let $g:\mathscr{C}\to\mathscr{A}$ be a mapping defined as $g(C_B)=f(B)$. We shall show that $\mathscr{C}$ is cushioned in $\mathscr{A}$ with cushion map g. If $x\in X$, then $x\in C_B$ where B is the smallest $B\in\mathscr{B}$ containing x. This shows that $\mathscr{C}$ is a refinement of $\mathscr{A}$. To prove that $\mathscr{C}$ is cushioned in $\mathscr{A}$, let $\mathscr{C}'$ be any subfamily of $\mathscr{C}$ and let $y\in\overline{\cup\{C_B : C_B\in\mathscr{C}'\}}$. Since $\mathscr{B}$ is a covering of x, there exists $B\in\mathscr{B}$ such that $y\in B$. Then $B\cap C_{B'}=\phi$ for all $B'>B$. If now $\mathscr{C}''=\{C_{B'} : B'<B,\ C_{B'}\in\mathscr{C}'\}$, it can be easily verified that $y\in\overline{\cup\{C_{B'} : C_{B'}\in\mathscr{C}''\}}$.

Let $\mathscr{B}''=\{B'\in\mathscr{B} : C_{B'}\in\mathscr{C}''\}$. Then $B'<B$ for all $B'\in\mathscr{B}''$. Therefore, we have,

$Cl_B[(\cup\{B' : B'<B, B'\in\mathscr{B}''\})\cap B]=Cl_B[\cup\{B'\cap B : B'\in\mathscr{B}''\}]\subseteq\cup\{f(B') : B'\in\mathscr{B}''\}=\cup\{g(C_{B'}) : C_{B'}\in\mathscr{C}''\}\subseteq\cup\{g(C_{B'}) : C_{B'}\in\mathscr{C}'\}$. Since B is an open set, therefore we have,

$Cl_B[(\cup\{B' : B'\in\mathscr{B}''\})\cap B]=\overline{\cup\{B' : B'\in\mathscr{B}''\}}\cap B$. Now, $y\in B$ and $y\in\overline{\cup\{C_{B'} : C_{B'}\in\mathscr{C}''\}}\subseteq\overline{\cup\{B' : B'\in\mathscr{B}''\}}$. Thus $y\in\cup\{g(C_{B'}) : C_{B'}\in\mathscr{C}'\}$ and hence $\mathscr{C}$ is a cushioned refinement of $\mathscr{A}$. Therefore X is mildly-paracompact (cf. [3] Theorem 1.10. (b)).

Theorem 2. *For a normal space X, the following are equivalent:*

(a) *X is mildly-paracompact.*

(b) *Every countable, regular open covering of X has a strongly compact-finite, open refinement.*

(c) *Every countable, regular open covering of X has a compact-finite, open refinement.*

(d) *Every countable, regular open covering of X has a strongly cs-finite open refinement.*

(e) *Every countable, regular open covering of X has a cs-finite open refinement.*

Proof. (a) $\Longrightarrow$ (b). Let $\mathscr{G}$ be any countable, regular open covering of X. Since X is mildly-paracompact, there exists a locally-finite, open refinement $\mathscr{H}$ of $\mathscr{G}$. Then the family of closures of members of $\mathscr{H}$ is locally-finite. It is easily verified that every locally-finite family is compact-finite. This means that $\mathscr{H}$ is strongly compact-finite.

(b) $\Longrightarrow$ (c). Obvious.

(c) $\Longrightarrow$ (d). Let $\mathscr{A}$ be any countable, regular open covering of X. Then there exists a compact-finite, open refinement $\mathscr{B}=\{B_\alpha : \alpha\in\Lambda\}$ of $\mathscr{A}$. Now, $\mathscr{B}$ being compact-finite is also point-finite. Since X is normal, there exists another open covering $\mathscr{C}=\{C_\alpha : \alpha\in\Lambda\}$ of X such that $\overline{C}_\alpha\subseteq B_\alpha$ for all $\alpha\in\Lambda$. Since $\mathscr{B}$ is compact-finite therefore $\{\overline{C}_\alpha : \alpha\in\Lambda\}$ is also compact-finite and hence also cs-finite. Thus $\mathscr{C}$ is a strongly cs-finite open refinement of $\mathscr{A}$.

(d) $\Longrightarrow$ (e). Obvious.

(e) $\Longrightarrow$ (a). Let $\mathscr{A}$ be any countable, regular open covering of X. Let $\mathscr{B}=\{B_\alpha : \alpha\in\Lambda\}$ be cs-finite open refinement of $\mathscr{A}$. We shall show that $\mathscr{B}$ is point-finite. Let $x\in X$ and let A_x be the closure of the convergent, constant sequence $\langle x\rangle$. Then $x\in A_x$ and therefore since $\mathscr{B}$ is cs-finite, A_x can intersect at most finitely many members of $\mathscr{B}$. Thus every countable, regular open covering of X has a point-finite, open refinement and hence X is mildly-paracompact (cf. [3], Theorem 1.6 (c)).

Theorem 3. *If a normal space X has W-weak topology with respect to an F-hereditary collection $\mathscr{K}$ and if every countable, regular open covering of X has a $\mathscr{K}$-finite, closed refinement, then X is mildly-paracompact.*

Proof. Since every $\mathscr{K}$-finite, closed family in such a space is locally-finite (cf. [1], Lemma 2.1), therefore every countable, regular open covering of such a space will have a locally-finite, closed refinement and hence also a countable, locally-finite, closed refinement. Therefore X is mildly-paracompact (cf. [3], Theorem 1.8 (b)).

Corollary 1. *If a normal space X has the W-weak topology with respect to F-hereditary collection $\mathscr{K}$ and if every countable, regular open covering of X has a $\mathscr{K}$-finite, open refinement, then X is mildly-paracompact.*

Proof. Let $\mathscr{U}$ be any countable, regular open covering of X. Let $\mathscr{V}=\{V_\alpha : \alpha\in\Lambda\}$ be a $\mathscr{K}$-finite, open refinement of $\mathscr{U}$. Now $\mathscr{V}$ is $\mathscr{K}$-finite and X has W-weak topology with respect to $\mathscr{K}$. Therefore $\mathscr{V}$ is point-finite. Since X is normal, there exists an open covering $\mathscr{W}=\{W_\alpha : \alpha\in\Lambda\}$ of $\mathscr{V}$ such that $\bar{W}_\alpha\subseteq V_\alpha$ for each α. Then $\{\bar{W}_\alpha : \alpha\in\Lambda\}$ is a $\mathscr{K}$-finite, closed refinement of $\mathscr{U}$ and hence X is mildly-paracompact by the above theorem.

Corollary 2. *A normal, k space is mildly-paracompact iff every countable, regular open covering of X has a compact finite, closed refinement.*

Proof. Only the 'if' part need be proved. Since X is a k-space, therefore X has W-weak topology with respect to the F-hereditary collection of all compact sets and therefore every compact finite, closed family is locally-finite. The result now follows from the above theorem.

Corollary 3. *A normal, sequential space is mildly-paracompact iff every countable, regular open covering of X has a cs-finite, closed refinement.*

Proof. To prove the 'if' part, we need only observe that a space is a sequential space iff it has W-weak-topology with respect to the F-hereditary collection of all sets which are closures of convergent sequences in X.

Theorem 4. *A k-space is mildly-paracompact iff every countable, regular open covering of X has a strongly compact-finite open refinement.*

Proof. Only the 'if' part need be proved. Let $\mathscr{A}$ be any countable, regular open covering of X. Then there exists a strongly compact-finite, open refinement $\mathscr{B}$ of $\mathscr{A}$. Since X is a k-space, it has W-weak topology with respect to the F-hereditary collection of all compact subsets of X. Therefore, the family of closures of members of $\mathscr{B}$ being compact-finite, is locally-finite. Hence $\mathscr{B}$

is a locally-finite open refinement of $\mathscr{A}$ and X is therefore mildly-paracompact.

Theorem 5. *A locally-compact space X is mildly-paracompact iff every countable, regular open covering of X has a compact-finite open refinement.*

Proof. Every compact-finite family in a locally-compact space is locally-finite.

Theorem 6. *A sequential space is mildly-paracompact iff every countable, regular open covering of X has a strongly-cs-finite, open refinement.*

Proof. The 'only if' part is obvious. To prove the 'if' part, let $\mathscr{A}$ be any countable, regular open covering of X. Let $\mathscr{B}$ be a strongly cs-finite, open refinement of $\mathscr{A}$. Since X is sequential, it has W-weak topology with respect to the F-hereditary collection of all closures of convergent sequences. Therefore the family of closures of members of $\mathscr{B}$ is locally-finite and hence $\mathscr{B}$ is a locally-finite, open refinement of $\mathscr{A}$. This proves that X is mildly-compact.

Theorem 7. *A first-axiom space is mildly-paracompact iff every countable, regular open covering of X has a cs-finite, open refinement.*

Proof. Every cs-finite family in a first-axiom space is locally-finite (cf. [1], Lemma 3.9).

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REFERENCES

[1] J. R. Boone: *Some characterisations of paracompactness in k-spaces.* To appear.

[2] Asha Rani Singal: *On countably-paracompact spaces.* To appear.

[3] M. K. Singal and Shashi Prabha Arya: *On mildly-paracompact spaces.* Bull. Aust. Math. Soc. 4 (1971), 273-277.

[4] ———: *A note on order paracompactness.* Annales de la Société Scientifique de Bruxelles, 84 (1970), 21-35.

[5] H. Tamano: *A characterisation of paracompactness.* To appear.

[6] J. E. Vaughan: *Linearly ordered collections and paracompactness.* To appear.

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