ON SURFACES IN 3-SPACE

By

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(Received September 16, 1970)

1. Introduction

The following considerations are based upon the semi-linear point of view. Throughout this paper a surface means a connected closed 2-submanifold in Euclidean 3-space E^3 unless otherwise stated. *J. Milnor* investigated the connected sum of connected orientable closed 3-manifolds and its prime decomposition [7]. In **2** of this paper we will define the isotopy sum of surfaces, substituting isotopy of E^3 for homeomorphic sense of *Milnor's* definition of connected sum, and its prime decomposition. The uniqueness of the prime decomposition for isotopy sum of surfaces is an open problem and seem too complicated. So we will examine mainly the special case, the surfaces of genus 2. It is the main purpose of this paper to prove the following theorems;

Theorem 1. (*existense theorem*) *Every nontrivial surface has a prime decomposition.*

Theorem 2. (*special case of uniqueness theorem*) *For any surface of genus* 2, *the prime decomposition is unique up to isomorphism.*

Decomposition of a surface is closely related to simple loops on the surface. Earlier, *Fox* [3] and *Homma* [5] proved the existense of a non-trivial *E*- or *I*-unknotted loop on any nontrivial surface. Recently *Waldhausen* [13] proved the uniqueness of the Heegaard splitting of 3-sphere S^3. His theorem (3, 1) [13] is a special case of the uniqueness of prime decomposition for bi-free surfaces. In **7** we will study with surfaces of genus $\geqq 2$ a little. At last we will give two prime surfaces as examples.

Two surfaces M and M' are said to be isomorphic, denoted by $M \approx M'$, if there is an isotopy of E^3 throwing M onto M'. Especially, a trivial surface means a surface isomorphic to the 2-sphere S^2. We denote the isomorphic class of a surface M by $[M]$, so M is a representative of $[M]$. But we will not distinguish the representative from its isomorphic class unless confusion.

I wish to thank Professor *T. Homma* for suggesting the problem and many discussion.

2. Isotopy sum

For a surface M we use $\mathrm{Int}\,M$ and $\mathrm{Ext}\,M$ to be denote the closures of bounded and unbounded components, respectively, of E^3-M. And define $\mathring{\mathrm{I}}\mathrm{nt}\,M=\mathrm{Int}\,M-M$ and $\mathring{\mathrm{E}}\mathrm{xt}\,M=\mathrm{Ext}\,M-M$. Two surfaces M_0 and M_1 are said to be *separated*, if there are disjoint 3-balls B_0 and B_1 such that $M_i\subset\mathring{B}_i$, $i=0, 1$. Equivalently above, there is a 3-ball B such that $M_0\subset\mathring{B}$ and $B\subset\mathring{\mathrm{E}}\mathrm{xt}\,M_1$ (or $M_1\subset\mathring{B}$ and $B\subset\mathring{\mathrm{E}}\mathrm{xt}\,M_0$).[1)]

Definition 1. *Let M_0 and M_1 be separated two surfaces and B be an associated 3-ball; $M_0\subset\mathring{B}$, $B\subset\mathring{\mathrm{E}}\mathrm{xt}\,M_1$. Let $h: D^2\times I\longrightarrow\mathrm{Ext}\,M_0\cap\mathrm{Ext}\,M_1$ be an embedding of 3-ball such that $h(D\times I)\cap M_i=h(D\times i)$, $i=0, 1$, and $h(D\times I)\cap\dot{B}=h(D\times 1/2)$.*[2)]

We call the surface $M_0\# M_1$ the *isotopy sum* of surfaces M_0 and M_1, defined by

$$M_0\# M_1=M_0\cup M_1\cup h(\dot{D}\times I)-h(\mathring{D}\times\dot{I}) .$$

Let note that

$$M_0\approx[(M_0\# M_1)\cap B]\cup[\mathrm{Int}\,(M_0\# M_1)\cap\dot{B}]=\mathrm{bd}[\mathrm{Int}\,(M_0\# M_1)\cap B] ,$$

$$M_1\approx[(M_0\# M_1)-B]\cup[\mathrm{Int}\,(M_0\# M_1)\cap\dot{B}]=\mathrm{bd}[\mathrm{Int}\,(M_0\# M_1)-\mathring{B}] .$$

Obviously $M_0\# M_1$ does not depend on order of M_0 and M_1; $M_0\# M_1\approx M_1\# M_0$, and the associated 3-ball B. From the homogeneity of manifold [4] and from that $h(D\times I)\cap\dot{B}=h(D\times 1/2)$, $M_0\# M_1$ is independent from the choice of h up to isomorphism. So, the above isotopy sum of surfaces is well defined up to isomorphism. Hence, $[M_0\# M_1]=[M_0]\#[M_1]$ is also well defined. The sum operation $\#$ is commutative, associative and trivial surface serves as identity; $[M]=[M]\#[S^2]$.

We call $M\approx M_1\# M_2\#\cdots\# M_k$ *a decomposition of M into factors M_j, $j=1,\cdots,k$.* A nontrivial decomposition means a decomposition of which each factor is non trivial.

Definition 2. *A non-trivial surface M is said to be prime, if either M_1 or M_2 is trivial for any decomposition $M\approx M_1\# M_2$ of M. A prime decomposition means a decomposition of which each factor is prime.*

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1) $\mathrm{bd}\,A$ and $\dot{A}$=the boundary of A,
$\mathrm{int}\,A$ and $\mathring{A}$=the interior of A, and
$\mathrm{cl}\,A$ and $\bar{A}$=the closure of A, throughout this paper.

2) I means a closed unit interval; $I=[0, 1]$.

Lemma (2, 1). *For any decomposition $M \approx M_1 \# M_2$ of M, $g(M)=g(M_1)+g(M_2)$, where $g(M)$ is the genus of M.*

Corollary to Lemma (2, 1). *Any surface of genus 1 is prime.*

Proofs of above are trivial and we drop them.

Proof of Theorem 1. If a nontrivial surface M is non prime, then there exists a nontrivial decomposition $M \approx M_1 \# M_2$. And if either M_1 or M_2 is non prime, one can decompose M to $M \approx M_1' \# M_2' \# M_3'$, and so on. By lemma (2, 1), $g(M)=g(M_1')+g(M_2')+g(M_3')$. Then from above corollary this process must terminate after a finite number $<g(M)$ of steps.

Remark. In this paper we disregard the orientation of surfaces in E^3. Considering an oriented 2-submanifold M in S^3 (ofcourse M is orientable if $M \subset S^3$), the orientation of M is determined by appointing one of components of S^3-M. Then isomorphism and isotopy sum of surfaces in S^3, similarly, could be defined as in E^3. But in this situation we must careful with respect to the position of two surfaces in S^3 from which we will construct the isotopy sum of them. That is, separated condition in S^3 is defined by adding to the same in E^3 that two appointing components (with respect to the orientation) of them are set disjoint. Further, for oriented surfaces M_1 and M_2 in oriented closed 3-manifolds N_1 and N_2, respectively, the sum of them can be also defined by "relative connected sum" of 3-manifolds. For more precise, see [13].

3. Simple loops on surfaces.

A loop (a simple closed polygonal curve) J on a surface M is said to be E-unknotted (I-unknotted), if there exists a proper 2-disk in $\operatorname{Ext} M(\operatorname{Int} M)$ which is bounded by J.[3] By *Dehn's* lemma, if $J \simeq 1$ in $\operatorname{Ext} M(\operatorname{Int} M)$ then J is an E-unknotted (I-unknotted) loop. We say J a bi-unknotted loop if both E- and I-unknotted. And a loop J is trivial on M if $J \simeq 1$ on M.[4]

Lemma (3, 1). *If a loop J on a surface M is bi-unknotted, then $J \sim 0$ on M.*[5]

Proof. From the definition, there are two proper disks D_1 and D_2 in $\operatorname{Ext} M$ and $\operatorname{Int} M$, respectively, such that $\dot{D}_1=\dot{D}_2=J$. $D_1 \cup D_2$ is a polyhedral 2-sphere in E^3. $\mathring{\operatorname{Int}}(D_1 \cup D_2)$ and $\mathring{\operatorname{Ext}}(D_1 \cup D_2)$ are separated in E^3 by $(D_1 \cup D_2)$. Since

3) M is proper in N if $\dot{N} \cap M=\dot{M}$.

4) $\simeq$ means homotopic to, ($\simeq 1$ means null homotopic),

5) $\sim$ means homologue to, (~ 0 means null homologues),

M is orientable J has a bi-collar neighborhood V on M such that $(V, J)\cong(S^1\times I, S^1\times 1/2)$.[6] J separates $V-J$ into $V_0=(S^1\times[0, 1/2))$ and $V_1=(S^1\times(1/2, 1])$. We may assume that $V_0\subset\mathring{\mathrm{I}}\mathrm{nt}(D_1\cup D_2)$ and $V_1\subset\mathring{\mathrm{E}}\mathrm{xt}(D_1\cup D_2)$. Suppose $J\not\sim 0$ on M(i.e. J does not separate M), then any pair of points $p_0\in V_0$ and $p_1\in V_1$ are joined by an arc A on M which does not intersect with J. Then $(D_1\cup D_2)\cap A\neq\phi$, but $(D_1\cup D_2)\cap M=J$. This contradicts to $J\cap A=\phi$.

Lemma (3, 2). *For any non-trivial surface M, the following statements are equivalent;*

(1) *M is prime, and*

(2) *any bi-unknotted loop on M is trivial on M.*

Proof. (2)⟶(1) is trivial from the definition 1, then we will show only (1)⟶(2). Suppose there is a non-trivial bi-unknotted loop J on M. Then there exist proper 2-disks D_1 and D_2 in $\mathrm{Ext}\, M$ and $\mathrm{Int}\, M$, respectively, such that $D_1\cap D_2=\dot{D}_1=\dot{D}_2=J$. By lemma (3, 1) $J\sim 0$ on M. Let $f: D^2\times I\longrightarrow\mathrm{Int}\, M$ be an embedding such that $f(D\times 1/2)=D_2$, $f(\dot{D}\times I)\subset M$ and $f(\mathring{D}\times I)\subset\mathring{\mathrm{I}}\mathrm{nt}\, M$. And let $M_1=\mathrm{bd}\,[\mathrm{Int}\, M-f(D\times\mathring{I})]\cap[\mathring{\mathrm{I}}\mathrm{nt}\,(D_1\cup D_2)]$, and $M_2=\mathrm{bd}\,[\mathrm{Int}\, M-f(D\times\mathring{I})]-M_1$. Since J is nontrivial on M, both surfaces M_1 and M_2 are nontrivial. Put $\mathrm{Int}\,(D_1\cup D_2)=B^3$. Then $f(D\times I)\cap\dot{B}^3=f(D\times 1/2)$. The conditions of definition 1 are satisfied; $M\approx M_1\# M_2$. Hence M is non prime, and this completes the proof.

Definition 3. Let L_1 and L_2 be two loops on a surface M. We may assume that $(L_1\cap L_2)$ consist of finite number of points and that L_1 and L_2 are crossing each other at each point of $L_1\cap L_2$. Let denote the number of points of $(L_1\cap L_2)$ by $n(L_1, L_2)$. *A pair of loops L_1 and L_2 are said to be normal on M (or, in normal position on M), if $n(L_1, L_2)\leqq n(L_1, h_1(L_2))$ for any isotopy $h_t(0\leqq t\leqq 1)$ of M.*[7] If L_1 and L_2 are normal pair of loops on M, then L_2 and L_1 are also normal pair on M.

4. Some lemmas

Lemma (4, 1). *For any surface M of genus n, $H_1(\mathrm{Ext}\, M)\cong H_1(\mathrm{Int}\, M)$ is isomorphic to free abelian group with n bases.*

Proof. From [3], $\mathring{\mathrm{I}}\mathrm{nt}\, M$ is homeomorphic to the complement of some solid torus of genus n in S^3. And lemma is obtained by the Alexander duality.

6) $\cong$ means homeomorphic to, or group isomorphism in later.

7) Throughout this paper, isotopy $h_t(0\leqq t\leqq 1)$ means such that $h_0=1$ (identity).

Lemma (4, 2). *Let N^3 be a compact 3-manifold in E^3 with connected boundary $\dot{N}=M$ of genus$\geqq 2$. Suppose the homomorphism $i^*: \pi_1(M) \longrightarrow \pi_1(N)$, induced by the inclusion $i: M \longrightarrow N$, has a non-trivial kernel. Then there exists a proper 2-disk D in N with non-trivial boundary $\dot{D}$ on M such that $D \sim 0$ on M.*

Proof. From the loop theorem [8], there exists a nontrivial simple loop on M which is null homotopic in N. By *Dehn's* lemma [9] it bounds a proper 2-disk D in N. If $D \not\sim 0$ on M, we can construct another disk D' in N satisfying the required properties from D.

Corollary (4, 3). *For any surface M of genus $\geqq 2$, there exists an E- or I-unknotted non-trivial loop J on M such that $J \sim 0$ on M.*

Corollary (4, 3) is obtained from (4, 2) and that for any surface of genus $\geqq 1$ there is an E- or I- unknotted non-trivial loop on M [3] [5].

Lemma (4, 4). *Let N^3 be a compact 3-manifold in E^3 with connected boundary of genus n. If $\pi_1(N)$ is a free group then N is a solid torus of genus n.*

Proof of (4, 4) is trivial by induction on genus of the boundary surface from (4, 1) and (4, 2).

Lemma (4, 5). *Suppose h is an isotopy of a surface M, then there is an isotopy H of E^3 which is an extension of h.*

Proof. We can decompose h into a finite number of isotopies of M of which each factor is supported by a 2-disk. And extends each isotopy to the isotopy of E^3 supported by a regular neighborhood of a 2-disk. Required H is a product of them.

Lemma (4, 6). *Let D_1 and D_2 be proper 2-disks in* Ext M *(or* Int M*). If $\dot{D}_1$ and $\dot{D}_2$ are normal on M, then there is a proper 2-disk D_1' in* Ext M *(or* Int M*) such that $\dot{D}_1'=\dot{D}_1$ and $(D_1' \cap D_2)$ consist of $1/2\,n(\dot{D}_1, \dot{D}_2)$ proper arcs. In the case that D_1 and D_2 are in* Int M*, we can choose D_1' isotopic to D_1 in* Int M *(so, in E^3) keeping M fixed.*

Proof. We may assume that $D_1 \cap D_2$ consists of a finite number of disjoint simple loops $L_1, L_2, \cdots, L_k$ and $1/2\,n(\dot{D}_1, \dot{D}_2)$ disjoint proper arcs. One of loops, say L_1, must bound a 2-disk D_0 in $\mathring{D}_2$ such that $D_0 \cap D_1 = \dot{D}_0 = L_1$. On the other hand, there is a 2-disk, say D_0', in $\mathring{D}_1$ bounded by L_1. Let $\hat{D}_1' = (D_1 - D_0') \cup D_0$, and deform slightly away from D_2. One obtains a proper 2-disk D_1^* such that (1) $\dot{D}_1^* = \dot{D}_1$, (2) loops in $D_1^* \cap D_2$ are some of $L_2, L_3, \cdots, L_k$, and (3) arcs in $D_1^* \cap D_2$

are the same of $D_1\cap D_2$. In the above step, if $\mathring{\text{E}}\text{xt}(D_0\cup D_0')\supset M$, we will show that D_1^* could be taken isotopic to D_1 in E^3 keeping M fixed. Let $N=N(\text{Int}(D_0\cup D_0'); E^3)$ be a regular neighborhood of $\text{Int}(D_0\cup D_0')$ in E^3 (it is sufficient that N is a 2-nd derived neighborhood of $\text{Int}(D_0\cup D_0')$ in E^3 for some trianguration of E^3 with M, D_1, D_2, etc. subcomplexes). A connected component $\hat{D}_0$ of $N\cap D_2$ which contains D_0 is a proper 2-disk in N. $\hat{D}_0$ separates N into two 3-balls N_1 and N_2; $N_1\cup N_2=N$, $N_1\cap N_2=\hat{D}_0$. $\text{Int}(D_0\cup D_0')$ is in either N_1 or N_2, say in N_1. Let $h_t(0\leqq t\leqq 1)$ be an isotopy of N such that $h_t|\dot{N}=1$ and $h_1[\text{Int}(D_0\cup D_0')]\subset N_2-\hat{D}_0$. Extend h_t to an isotopy $H_t(0\leqq t\leqq 1)$ of E^3 by $H_t|N=h_t$ and $H_t|E^3-N=1$. Then $H_1(D_1)$ satisfies the conditions (1)-(3) of the above as D_1^*. Repeating the above process, one obtains 2-disk D_1' such that $\dot{D}_1'=\dot{D}_1$ and $D_1'\cap D_2$ contains no loops. This completes the proof.

Lemma (4, 7). *Suppose $M\approx M_1\# M_2$ is a decomposition of M, then*

$\pi_1(\text{Ext}\,M)=\pi_1(\text{Ext}\,M_1)*\pi_1(\text{Ext}\,M_2)$, *and*

$\pi_1(\text{Int}\,M)=\pi_1(\text{Int}\,M_1)*\pi_1(\text{Int}\,M_2)$, *where $*$ means free product of groups.*

Proof of this lemma is trivial from definition 1 and by the *van. Kampen* theorem.

Definition 4. *A surface M is said to be E-free (I-free), if $\pi_1(\text{Ext}\,M)\pi_1(\text{Int}\,M)$ is a free group of rank>0. We say a surface M bi-free if both E- and I-free.*

Suppose a surface M of genus 1 is in S^3, then the closure of one of components of S^3-M is a solid torus [1]. From this the following lemma is trivial.

Lemma (4, 8). *For any surface M of genus 1, one of the following three different cases arises;*

(1) *M is bi-free,*

(2) *M is I-free and $\pi_1(\text{Ext}\,M)\cong$a knot group $\not\cong Z$, and*

(3) *M is E-free and $\pi_1(\text{Int}\,M)\cong$a knot group $\not\cong Z$.*

Lemma (4, 9). *Any nonprime surface M of genus 2 fall in one of the following different 6 cases (i.e. for a non-trivial decomposition $M\approx M_1\# M_2$ of M, we have the following table by reordering indices).*[8]

	M_1	M_2	$\pi_1(\text{Int}\,M)$	$\pi_1(\text{Ext}\,M)$
(1)	bi-free	bi-free	$Z*K$	$Z*Z$
(2)	bi-free	E-nonfree	$Z*Z$	$Z*K$
(3)	bi-free	I-nonfree	$Z*K$	$Z*Z$
(4)	E-nonfree	E-nonfree	$Z*Z$	K_1*K_2
(5)	I-nonfree	I-nonfree	K_1*K_2	$Z*Z$
(6)	E-nonfree	I-nonfree	$Z*K$	$K'*Z$

8) in the table, each of K, K', K_i means any knot group but Z (infinite cyclic).

Proof. It follows from (4, 7) and (4, 8) and by that any knot group is indecomposable with respect to the free product of group [9].

Lemma (4, 10). *Any two bi-free surfaces M and M′ of genus 1 are isomorphic.*

Proof. By (4, 4) Int M is a solid trous of genus 1. Then there is a polygonal loop L in $\mathring{\text{I}}\text{nt}\, M$ such that Int M is a regular neighborhood of L in E^3. $\pi_1(E^3-L) \cong \pi_1(\text{Ext}\, M) \cong Z$. By Dehn's lemma L is a trivial knot in E^3. Similarly for M', there is a loop L' of which Int M' is a regular neighborhood in E^3. Hence there exists an isotopy of E^3 throwing L onto L'. From the uniqueness of the regular neighborhoods this gives an isomorphism of M to M'.

By (4, 10), there may be no confusion if we denote a bi-free surface of genus 1 by T. And we also denote $mT \approx T_1 \# T_2 \# \cdots \# T_m$, where each T_i is a surface isomorphic to T.

5. Existence of prime surfaces.

Theorem. (*Suzuki*) [11] [12]. *For any integer $n \geqq 1$, there exists a prime surface of genus n.*

Suzuki [11] [12] constructed so complicated surfaces in S^3 which are extension of Homma's example [5]. In his paper Suzuki defined the primeness of the surfaces in S^3 by the property (2) in our lemma (3, 2). Through the natural inclusion $n: E^3 \longrightarrow S^3$, where $S^3 - n(E^3) = \infty$ is an infinite point of E^3, primeness of the surface in E^3 and in S^3 are equivalent. We denote the one-point compactification of Ext M by $\widetilde{\text{E}}\text{xt}\, M = S^3 - n(\mathring{\text{I}}\text{nt}\, M)$ for any surface M in E^3. Here we will show only the primeness of the *Homma's* example H of genus 2.

Example 1. *The surface H (in Figure 1) of genus 2 is prime.*

Proof. At first it is easily cheked that $\pi_1(\text{Int}\, H) \cong K * K$, where K is a knot

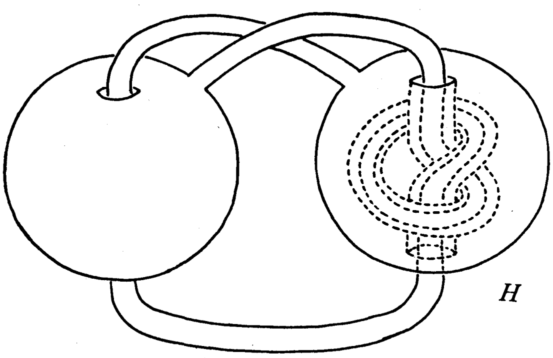

Figure 1.

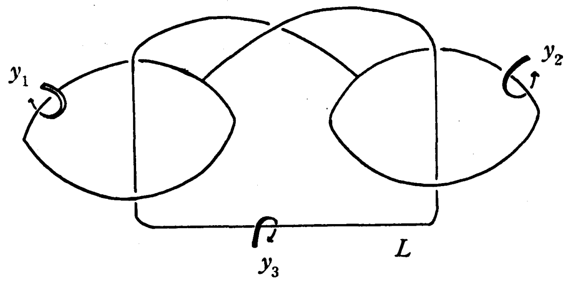

Figure 2.

group of the clover leaf. By (4, 10) we prove H prime if we check that $\pi_1(\mathrm{Ext}\,H) \ncong Z*Z$. It is obvious that $\mathrm{E}\overset{\circ}{\mathrm{x}}\mathrm{t}\,H \cong E^3-L$, where L is a connected linear graph in Figure 2. Calculdting $\pi_1(E^3-L)$ by the way of *Kinoshita* [6], we obtain that;

$\pi_1(E^3-L) \cong (y_1, y_2, y_3;\ y_1y_3y_1^{-1}y_3^{-1}y_2y_3y_2^{-1})$, and 2-nd Alexander polynomial $\Delta_L^{(2)}(t) = 2t-1$, where generators y_1, y_2, and y_3 are as in Figure 2. Hence $\pi_1(\mathrm{Ext}\,H)$ is indecomposable with respect to free product (then there exists no nontrivial E-unknotted loop on H), and H is prime.

6. Proof of Theorem 2.

Since we have the rough classification of non-prime surfaces of genus 2 by lemma (4, 9), then we complete the proof of the theorem if we check the 6 cases in (4, 9).

(6, 1) For the case (1) in (4, 9), the theorem follows directly from (4, 10).

Let $M \approx M_1 \# M_2$ be a non-trivial decomposition (so, prime decomposition) of a given non prime surface M of genus 2 with an associated 3-ball B^3. Suppose $M \approx M_1' \# M_2'$ is another non-trivial decomposition of M with an associated 3-ball B'. By the definition 1, we may assume that $M_1=(M\cap B)\cup C$, $M_2=(M-\overset{\circ}{B})\cup C$, $M_1'=(M\cap B')\cup C'$ and $M_2'=(M-\overset{\circ}{B'})\cap C'$, where $C=\dot{B}\cap \mathrm{Int}\,M$ and $C'=\dot{B}'\cap \mathrm{Int}\,M$. Let denote $\dot{B}-\overset{\circ}{C}=D$ and $\dot{B}'-\overset{\circ}{C}'=D'$. We may assume also that $\dot{D}$ and $\dot{D}'$ are in normal position on M and that $(D\cap D')$ and $(C\cap C')$ have no loops by (4, 5) and (4, 6).

(6, 2) For the cases (4) and (5) in (4, 9). The proof for the case (5) is similar to one for the case (4). So we will prove only for the case (4). For this case, we will assert that $\dot{B}\cap\dot{B}'=\phi$. For, if $\dot{B}\cap\dot{B}'\neq\phi$, $D\cap D'$ consists of finite union $\mathscr{J}$ of disjoint proper arcs $J_1, J_2, \cdots, J_k$. $\mathscr{J}$ separates D' into interior disjoint $k+1$ disks $D_1', D_2', \cdots, D_{k+1}'$ and bd $\mathscr{J}$ separates $\dot{D}'$ into $2k$ arcs $A_1', A_2', \cdots A_{2k}'$. Put two families $\mathscr{D}'=\{D_1', D_2', \cdots, D_{k+1}'\}$ and $\mathscr{A}'=\{A_1', A_2', \cdots, A_{2k}'\}$. Then there must be one disk, say D_1', such that $\dot{D}_1'\cap\dot{D}'$ is connected. So, there are arcs in $\dot{D}_1'$, say $J_1\subset\mathscr{J}$ and $A_1'\in\mathscr{A}'$, such that $\dot{D}_1'=J\cup A_1'$, $J_1\cap A_1'=\dot{J}_1=\dot{A}_1'$ and $(D_1'-J_1)\cap D=\phi$. On the other hand J_1 separates D into two disks D_1 and D_2 such that $D_1\cap D_2=\dot{D}_1=\dot{D}_2=J_1$, $D_1\cup D_2=D$ and $D_i\cap D_1'=\dot{D}_i\cap\dot{D}_1'=J_1$, $i=1, 2$. Put $A_i=\dot{D}_i-\overset{\circ}{J}_1$, $i=1, 2$. Note that $A_1\cup A_2=\dot{D}$ and $A_1\cap A_2=\dot{A}_1=\dot{A}_2$. $A_1'\cup A_i$, $i=1, 2$, are simple loops on either M_1 or M_2, say on M_1. $A_1'\cup A_i$ bounds a proper 2-disk $(D_1'\cup D_i)$ in $\mathrm{Ext}\,M$, $i=1, 2$. The E-unknotted loop $A_1'\cup A_i$ on E-nonfree surface M_1 of genus 1 must bound a 2-disk on M_1, $i=1, 2$. Then either $A_1'\cup A_1$ or $A_1'\cup A_2$

bounds a 2-disk E^2 on $M_1-\mathring{C}(\subset M)$. This contradicts to the normality of the pair of loops $\dot{D}$ and $\dot{D}'$. Hence $\dot{B}\cap\dot{B}'=\phi$. Then $\dot{D}'$ is in either $(M_1-\mathring{C})$ or $(M_2-\mathring{C})$. From that $\dot{D}'$ is non-trivial on $M, \dot{D}'$ is isotopic to $\dot{D}$ on M. Hence $M_1\approx M'_i$ and $M_2\approx M'_j$, $i\neq j$, $i, j=1, 2$.

(6, 3) For the cases (2) and (6) in (4, 9). We may assume that M_1 and M'_1 are E-free and M_2 and M'_2 are E-nonfree by (4, 8) and (4, 9). If $\dot{D}\cap\dot{D}'=\phi$, the theorem follows from the same as in (6, 2). So, suppose that $\dot{D}$ and $\dot{D}'$ are in normal position on M and $D\cap D'$ contains only finite number $\neq 0$ of proper arcs. Since M_1 is E-free of genus 1, there is an unique proper 2-disk E^2_0 in Ext M_1, up to isotopy of Ext M_1, such that $\dot{E}_0\subset M_1-\mathring{C}$ and $\dot{E}_0\not\sim 0$ an M_1. We may assume also that $E_0\cap D=\phi$ and $\dot{E}_0$ and $\dot{D}'$ are in normal position on M. Under this condition we will prove next;

(6, 4) $E_0\cap D'=\phi$.

For, if $E_0\cap D'\neq\phi$, we may assume that $E_0\cap D'$ contains only a finite number of proper arcs by (4, 6). Let $N_0=N(E_0;\ \text{Ext}\,M)$ be a small regular neighborhood of E_0 in Ext M such that $N_0\cap M=\dot{N}_0\cap(M_1-\mathring{C})\cong(S^1\times I)$ is a regular neighborhood of $\dot{E}_0$ in M. $(\overline{\dot{N}_0-M})$ consists of two isotopic 2-disks F_1 and F_2 in Ext M. $(F_i, F_i\cap D')\cong(E_0, E_0\cap D')$. Then $D'\cap(F_1\cup F_2)$ consists of finite union $\mathscr{K}=(K_1\cup K_2\cup\cdots\cup K_{2r})$ of disjoint $2r$ arcs. Let $Q=\text{bd}(\text{Int}\,M\cup N_0)$, then $Q\approx M_2$. $\mathscr{K}$ separates D' into a family $\mathscr{D}''=\{D''_1, D''_2,\cdots, D''_{2r+1}\}$ of interior disjoint $2r+1$ disks. And bd$\mathscr{K}$ separates $\dot{D}'$ into a family $\mathscr{A}''=\{A''_1, A''_2,\cdots, A''_{4r}\}$ of arcs. Then there must be one disk and two arcs as in (6, 2), say $D''_1\in\mathscr{D}''$, $K_1\subset\mathscr{K}$ and $A''_1\in\mathscr{A}''$ such that $\dot{D}''_1=K_1\cup A''_1$ and $K_1\cap A''_1=\dot{K}_1=\dot{A}''_1$. K_1 is in either F_1 or F_2, say in F_1. Obviously, D''_1 is a proper disk in Ext Q, then E-unknotted loop $\dot{D}''_1=K_1\cup A''_1$ must bound a 2-disk U_1 on Q. If $\mathring{U}_1\not\supset F_2$, it contradicts to that the pair of loops $\dot{D}'$ and $\dot{E}_0$ are normal on M. Then $\mathring{U}_1\supset F_2$. Hence there is a disk, say D''_2, in $\mathscr{D}''$ such that $D''_2\subset N_0$ and $D''_2\cap F_i=K_i\subset\mathscr{K}$, $i=1, 2$. And there exists a disk $D''_3\in\mathscr{D}''$ proper in Ext Q such that $D''_3\cap F_2=\dot{D}''_3\cap F_2\supset K_2$. $\dot{D}''_3$ must bound a disk U_3 on Q as $\dot{D}''_1$. Since $\dot{D}''_1\cap\dot{D}''_3=\phi$, $U_3\subset\mathring{U}_1$. If $\dot{D}''_3=\dot{U}_3$ does not intersect with F_1, it contradicts to the normality of the pair of loops $\dot{D}'$ and $\dot{E}_0$ in M, again. Then there is an arc, say $K_3\subset\mathscr{K}$ in $F_1\cap\dot{D}''_3$, and so on. This contradicts to the finiteness of elements of $\mathscr{K}$. Hence (6, 4) is proved.

Since M'_2 is E-nonfree of genus 1, $\dot{E}_0\subset M'_1$. So, we may assume that $N_0\cap M=\dot{N}_0\cap M\subset(M'_1-C')$. Hence $\{(M-N_0)\cup F_1\cup F_2\}=Q\approx M_2\approx M'_2$. And it is easily checked that all $A'_j \in \mathscr{A}'$ in $M_1 - \mathring{C}$ are isotopic relative $\dot{D}$ on M_1. From this and by the same argument as above for C and C' in Int M, one obtains that $M \approx M'_1$. This completes the proof of (6, 3).

(6, 5) It remains the proof of the theorem for the case (3) in (4, 9). This is entirely similar to (6, 3), except the order of the arguments for D, D' in Ext M and C, C' in Int M. Then the proof is completed.

Corollary (6, 6). *Suppose either* $\pi_1(\text{Ext}\, M) \cong G_1 * G_2$, *or* $\pi_1(\text{Int}\, M) \cong G_3 * G_4$ *for a surface* M, *where* G_i *is an indecomposable group with respect to free product and* $G_i \not\cong Z$, $i=1, 2, 3, 4$. *Then the prime decomposition of* M *is unique up to isomorphism.*

Proof of (6, 6) is the same as (6, 2).

7. Surfaces of genus $\geqq 2$

Waldhausen proved the uniqueness of the *Heegaard*-splitting of 3-sphere S^3 [13]. In other words, his theorem [13, (1, 3)] asserts that;

Theorem. (*Waldhausen*). *For any bi-free surface* M *of genus* $m>0$, $M \approx mT$.[9)]

And also the proofs of the theorem [13, (3, 1)] ensure that;

Theorem 3. *If* $M_1 \# mT \approx M_2 \# mT$ *for two surfaces* M_1, M_2 *and some integer* $m \geqq 1$, *then* $M_1 \approx M_2$.

Analogous argument as (6, 2) will lead us the next;

Theorem 4. *If a surface* M *has a non-trivial decomposition* $M \approx M_1 \# M_2 \# \cdots \# M_m$, *where each of* $\pi_1(\text{Int}\, M_i)$ (*or each of* $\pi_1(\text{Ext}\, M_i)$), $i=1, 2, \cdots, m$, *is indecomposable with respect to free product and not infinite cyclic, then the prime decomposition of* M *is unique up to isomorphism.*

Lemma (7, 1). *Suppose a surface* M *has a non-trivial decomposition* $M \approx M_1 \# M_2 \# \cdots \# M_k \# \cdots \# M_m$, *where* M_i *is* I-*nonfree of genus* 1 *if* $1 \leqq i \leqq k$ *and* E-*nonfree of genus* 1 *if* $k+1 \leqq i \leqq m$. *Let* $D_j, j=1, 2, \cdots, k$, *be any set of disjoint proper 2-disks in* Ext M *such that* $\{\dot{D}_j\}$ *are homologicaly independent on* M. *Then* $bd\{\text{Int}\, M \cup \bigcup_{j=1}^{k} N(D_j;\ \text{Ext}\, M)\} \approx M_{k+1} \# M_{k+2} \# \cdots \# M_m$, *where* $N(D_j;\ \text{Ext}\, M)$ *is a regular neighborhood of* D_j *in* Ext M, $j=1, 2, \cdots, k$.

This lemma (7, 1) is elementally proved by the way used in this paper before. Then from theorems 3, 4 and (7, 1) we will obtain easily that;

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9) see (4, 10).

Theorem 5. *If a surface M has non-trivial decompositions* $M \approx M_1 \# M_2 \# \cdots \# M_m$ *and* $M \approx M'_1 \# M'_2 \# \cdots \# M'_m$, *where* $m = g(M)$ *is the genus of M, then these two prime decmpositions of M coincide up to order and isomorphism.*

The author guess the next statement which is a generalization of the theorem (*Waldhausen*), but yet proved even for $n=2$.

Conjecture (7, 2). *If both* $\pi_1(\mathrm{Int}\, M) \cong A_1 * A_2 * \cdots * A_m$ *and* $\pi_1(\mathrm{Ext}\, M) \cong B_1 * B_2 * \cdots * B_m$ *are non-trivial free products for a surface M, then if* $m = g(M)$, *M will be non prime.*

In (7, 2) if $g(M) > m$, then there is a counter example (Example 3).

Example 2. The *I*-free surface M_2 (in Figure 3) of genus 2 is prime, by the following. But the primeness of M_2 is not given by the way of example 1. $\pi_1(\mathrm{Ext}\, M_2)$ is presented by the form;

$$(x_1, x_2, x_3;\ x_2 x_1^{-1} x_2^{-1} x_1 x_3^{-1} x_1^{-1} x_2 x_1 x_2^{-1} x_3 x_1^{-1} x_2 x_1 x_2^{-1} x_3^{-1}),$$

where the generators x_1, x_2 and x_3 are represented as in Figure 2. From [10, Theorem 1], $\pi_1(\mathrm{Ext}\, M_2)$ is indecomposable respect to free product. Hence M_2 is prime and homomorphism $i^*: \pi_1(M_2) \longrightarrow \pi_1(\mathrm{Ext}\, M_2)$ has trivial kernel, where i^* is induced by the inclusion $i: M \longrightarrow \mathrm{Ext}\, M_2$.

Example 3. The bi-nonfree surface M_3 (in Figure 4) of genus 3 is constructed from M_3 of example 2. From the construction, it is easily checked that $\pi_1(\mathrm{Ext}\, M_3)$

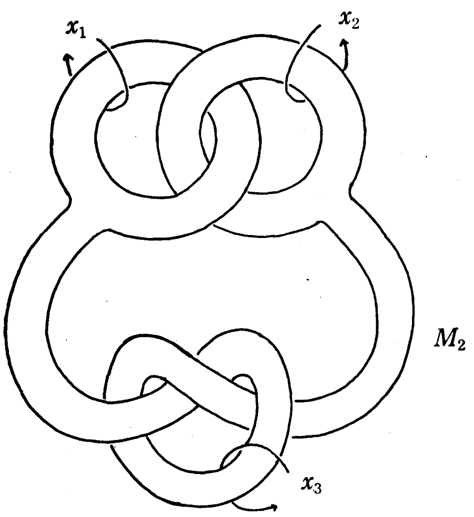

Figure 3.

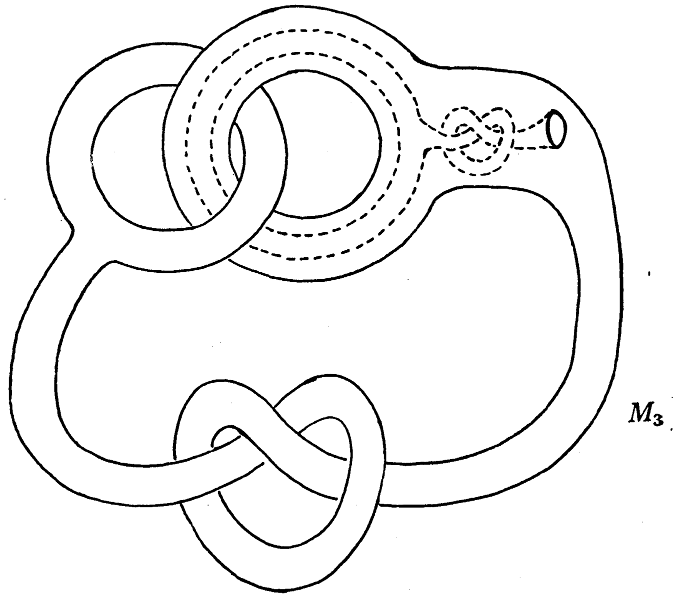

Figure 4.

$\cong\pi_1(\mathrm{Int}\, M_3)\cong\pi_1(\mathrm{Ext}\, M_2)*Z$ and M_3 is prime. Further, it is interest that $\tilde{\mathrm{E}}\mathrm{xt}\, M_3$ $\cong\mathrm{Int}\, M_3$, where $\tilde{\mathrm{E}}\mathrm{xt}\, M_3$ is a one-point compactification of $\mathrm{Ext}\, M_3$ (see 5). Note that if the conjecture (7, 2) is true for $n=2$ then there is no prime surface M of genus 2 such that $\tilde{\mathrm{E}}\mathrm{xt}\, M\cong\mathrm{Int}\, M$.

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